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Invariant metrics on Lie groups, and other Invariant structures on Homogeneous Spaces.

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Abstract

The theory of biharmonic maps is old and rich and has gained a growing interest in the last decade. The theory of harmonic maps into Lie groups, symmetric spaces or homogeneous spaces has been extensively studied in relation to integrable systems by many mathematicians. In particular, harmonic and biharmonic homomorphisms between Riemannian Lie groups (a Riemannian Lie group is Lie group endowed with left invariant Riemannian metric). In this thesis we discuss the study of biharmonic and harmonic homomorphisms between Riemannian Lie groups.

This dissertation concerns particularly harmonic and biharmonic homomorphisms between Riemannian Lie groups which is one of the topics studied on a Lie groups endowed with invariant structure. Various background material such as Lie groups, Invariant metrics, connections, curvatures, homogeneous spaces, harmonic homomorphism, biharmonic homomorphism, and representation theory are reviewed.

As a result, we classify biharmonic and harmonic homomorphisms $f : (G, g_1) \longrightarrow (G, g_2)$ where G is a connected and simply connected three-dimensional unimodular Lie group and g_1 and g_2 are left invariant Riemannian metrics.

ملخص

نظريَات التطبيقَات ثنَائِية التوَافقية قديمة و غنية مِمَا جعلهَا تكتسب إهتمَامَا متزَايدَا في العشر سنوَات المَاضية و كذلك نظريَات التطبيقَات التوَافقية بين زمر لي وَالفضَاءَات المتنَاظرة أَو المتجَانسة قد عرضت لَلبحث بصفة مكثفة بَالموَازَات مع الأَنظمة التكَاملية من طرف عدة بَاحثين في الريَاضيَات.

هذه الأَطروحة تنَاقش بصفة خَاصة التمَاثلَات التوَافقية و ثنَائِية التوَافقية بين زمر لي الريمَانية (زمر لي الريمَانية معنَاه زمرة لي مزودة بـ مسَافة ريمَانية ثَابتة من اليسَار) وَالتي تعتبر موضوع من أَهم الموَاضيع المدروسة حول زمر لي الريمَانية. لدرَاسة هذَا الموضوع إحتجنَا لتطلع علَى بحوث حول زمر لي الريمَانية، التغَاير حول زمر لي، الإنحنَاء، الفضَاءَات المتجَانسة، التمَاثلَات التوَافقية و ثنَائِية التوَافقية... إلخ.

نُنَاقش في هذه الأَطروحة النتَائج المتحصل عليهَا في مقَالنَا المعنون بـ < التمَاثلَات التوَافقية و ثنَائِية التوَافقية بين زمر لي ذَات بعد ٣، مترَابطة، مترَابطة بِبسَاطة و أُحَادية. > في هذَا المقَال قمنَا بتصنيف التمَاثلَات التوَافقية و ثنَائِية التوَافقية $f:(G,g_1)\longrightarrow(G,g_2)$ حيث G زمرة لي ذَات بعد ٣، مترَابطة، مترَابطة بِبسَاطة و أُحَادية و g_1, g_2 مسَافَات ريمَان ثَابتة من اليسَار.

Contents

Introduction

Generally the term of "Lie group" belongs to E. Cartan (1930). It is defined as a manifold G endowed with a group structure, such that the multiplication map and the inversion map are smooth (i.e.differentiable). The simple examples of Lie groups are the groups of isometries of $\mathbb{R}^n$, $\mathbb{C}^n$. Hence, we obtain the orthogonal group $O(n)$ and the unitary group $U(n)$. An algebra $\mathfrak{g}$ can be associated with each Lie group G in a natural way; this is called the Lie algebra of G.

The most important applications of Lie groups involve actions by Lie groups on other manifolds. A homogeneous space is a manifold M on which a Lie group acts transitively. As a consequense M is diffeomorphic to the coset space G/H where H is a subgroup of G. In fact, if we fix a point $m \in M$, then H is the isotropy subgroup of m. In many mathematical fields (geometry , harmonic analysis ...) a special interest is given to tensors, operators ... invariant on M. In the classical case of $M = \mathbb{R}$ these are simply the tensors, operators ... has constant coefficients.

The theory of biharmonic maps is old and rich and has gained a growing interest in the last decade (see [2, 11] and others). The theory of harmonic maps into Lie groups, symmetric spaces or homogeneous spaces has been extensively studied in relation to integrable systems by many mathematicians (see for examples [5, 12, 13]). In particular, harmonic maps of Riemann surfaces into compact Lie groups equipped with a bi-invariant Riemannian metric are called principal chiral models and intensively studied as toy models of gauge theory in mathematical physics [14]. In the papers [9, 6], harmonic inner automorphisms of a compact semi-simple Lie group endowed with a left invariant Riemannian metric where studied. In [3], there is a detailed study of biharmonic and harmonic homomorphisms between Riemannian Lie groups.[1]

[1] A biharmonic homomorphism between Riemannian Lie groups is a homomorphism of Lie groups $\phi : G \longrightarrow H$ which is also biharmonic where G and H are endowed with left invariant Riemannian metrics.

Chapter 1 contains a brief review of Riemannian manifolds, and a definition of group actions and describing examples. It contains also an introduction of proper actions, which give a nice properties to the quotients. The quotient manifold theorem gives conditions under which the quotient of smooth manifold is again smooth manifold. And then discusses a way to make a Lie group into a Riemannian manifold. The important metrics here are the bi-invariant metrics, with respect to such metrics we give formulas for the connection and the different types of curvatures. This chapter is concluded by giving the classification of left invariant metrics on simply connected three dimensional unimodular Lie groups.

The second chapter of this thesis gives a study of the harmonicity and bi-harmonicity of Riemannian Lie groups homomorphisms. The reference of This chapter is the article of M.boucetta and S. Ouakkas [3] , we start with a reminder of the elementary notions on Lie groups and left invariant metrics on a Lie group (See [10] for more details). By using this language we show that the harmonicity problem of a hoomorphism of Riemannian Lie groups is an algebraic problem; therefore, the study of the structure of the Lie algebras will be under question. The conditions of harmonicity of a homomorphism are expressed with respect to the structure of the Lie algebras in play. We consider the same problem by restricting the class of homomorphism to be studied, we consider successively the cases where the homomorphism is an automorphism, a Riemannian immersion, and a submersion. Finally, we study the situations where the harmonicity and the bi-harmonicity are equivalent.

Chapter 3 contains the obtained results in our article [1]. We classify, up to conjugation by automorphisms of Lie groups, harmonic and biharmonic homomorphisms $f:(G,g_1)\longrightarrow(G,g_2)$ where G is a non-abelian connected and simply-connected three dimensional unimodular Lie group, f is an homomorphism of Lie groups and g_1 and g_2 are two left invariant Riemannian metrics. There are five non-abelian connected and simply-connected three-dimensional unimodular Lie groups; the nilpotent Lie group Nil, the special unitary group SU(2), the universal covering group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the special linear group, the solvable Lie group Sol, and the universal covering group $\widetilde{\mathrm{E}_0}(2)$ of the connected component of the Euclidean group. Our main results are as follows:

1. For Nil and Sol we show that a homomorphism is biharmonic if and only if it is harmonic and we classify completely all the harmonic homomorphisms (see Theorems 3.1.1, 3.3.1 and 3.3.2).

2. For $\widetilde{\mathrm{E}_0}(2)$ we classify completely all the harmonic homomorphisms (see Theorem 3.2.1).

For this group there are biharmonic homomorphisms which are not harmonic and we give a complete classification of these homomorphisms (see Theorem 3.2.2). To our knowledge, these are the first examples of biharmonic not harmonic homomorphisms between Riemannian Lie groups.

3. For SU(2) and $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, we give a complete classification of harmonic homomorphisms (see Theorems 3.4.1 and 3.5.1). We show that these groups have biharmonic homomorphisms which are not harmonic and we give the first examples of these homomorphisms. For SU(2) we recover the results obtained in [9, 6] and we complete them.

Chapter 1

Preliminaries

Before we introduce the Lie groups and give examples about them, we first give a brief review of Riemannian manifolds. We mention the quotient manifold theorem which gives conditions under which the quotient of smooth manifold is again smooth manifold. Taking into consideration such metrics we give formulas for the connection and the different types of curvatures. We sum up by giving the classification of left invariant metrics on simply connected three dimensional unimodular Lie groups.

1.1 Review of the Riemannian manifolds

Definition 1.1.1. *A Riemannian metric on a smooth manifold M is a correspondence which associates to each point $p \in T_pM$ a scalar product $g_p = \langle , \rangle_p$ (that is a symmetric bilinear, positive definite form) on the tangent space T_pM, for any two smooth vector fields X, Y in a neighborhood of p, the map $p \longmapsto \langle X_p, Y_p \rangle_p$ is smooth. A smooth manifold with a Riemannian metric is called a Riemannian manifold, and is denoted $(M, g = \langle , \rangle)$.*

Let $(M, \langle , \rangle)$ be a Riemannian manifold and $(x_1, ..., x_n)$ local coordinates system on an open set U and let $\partial_{x_i} = \frac{\partial}{\partial x_i}$ (for $i = 1, n$) be the coordinate vector fields at p.
Then

$$g_p(X_p, Y_p) = \sum_{i=1}^{n} g_{ij}(p) X_p^i Y_p^j$$

where the local functions $g_{ij} : U \longrightarrow \mathbb{R}$ are given by $g_{ij}(p) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p$

The local expression of g is given by

$$g = \langle , \rangle = \sum_{i,j} g_{ij} dx_i dx_j$$

Where $dx_i dx_j = \frac{1}{2}(dx_i \otimes dx_j + dx_j \otimes dx_i)$
In the language of tensorsl, g is a symmetric, non-degenerate $(0,2)$ tensor field on M.

Proposition 1.1.1. *Any smooth manifold carries a Riemann metric.*

Proof. Choose a locally finite open covering $\mathbb{U} = \{U_\alpha\}_{\alpha \in A}$ of M by chart domains and a subordinate partition of the unity $(f_\alpha : M \longrightarrow [0,1])_{\alpha \in A}$. For any $\alpha \in A$ define on U_α a Riemannian metric $\langle , \rangle_\alpha$ by putting

$$\langle , \rangle_\alpha = \sum_{i=1}^{n} (dx_i)^2$$

Now define $\langle , \rangle$ on M by putting, for any $p \in M$ and any $u, v \in T_pM$,

$$\langle u, v \rangle = \sum_{\alpha \in A} f_\alpha \langle u, v \rangle_\alpha$$

One can see easily that $\langle , \rangle$ is a Riemannian metric on M □

Definition 1.1.2. *Let (M, g), (N, g') be Riemannian manifolds. An isometry is a diffeomorphism $f : (M, g) \longrightarrow (N, g')$ that preserves the metrics, i.e.*

$$g_p(u, v) = g'_{f(p)}(T_pf(u), T_pf(v)), \forall u, v \in T_pM$$

where $T_pf = d_pf$ is the tangent function.

Example 1. *(1) Let $M = \mathbb{R}^n$, Let $\{e_1, ..., e_n\}$ the canonical basis of $\mathbb{R}^n$ and $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, ..., 1..., 0)$ The metric is given by $g(e_i, e_j) = \delta_{ij}$. In this case $\mathbb{R}^n$ is called the Euclidean space of dimension n.*
(2) Let $f : M \longrightarrow N$ be an immersion (that is smooth, with d_pf one-to-one for all $p \in M$) let g' be a Riemannian metric on N, then f induces a Riemannian metric g on M by defining

$$g_p(u, v) = g'_{f(p)}(T_pf(u), T_pf(v)), \forall u, v \in T_pM$$

Let $\mathcal{X}(M)$ be the set of all smooth vector field on M and $\mathcal{F}(M)$ be the set of all smooth real-valued functions on a manifold M. Let $X, Y \in \mathcal{X}(M)$ Define $[X, Y] = XY - YX$. This is a function from $\mathcal{F}(M)$ to $\mathcal{F}(M)$ sending each f to $X(Yf) - Y(Xf)$. We can shows that $[X, Y]$ is a derivation on $\mathcal{F}(M)$, which is called the bracket of X and Y. The bracket gives to each $p \in M$ the tangent vector $[X, Y]_p$ such that

$$[X, Y]_p(f) = X_p(Yf) - Y_p(Xf)$$

We have, the bracket operation has the following properties:

1. $[X, Y] = -[Y, X]$ (skew-symmetry),
2. $[aX + bY, Z] = a[X, Z] + b[Y, Z], [Z, aX + bY] = a[Z, X] + b[Z, Y]$ ($\mathbb{R}$-bilinearity),
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

Definition 1.1.3. *An connection ∇ on a smooth manifold M is a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$, $(X, Y) \longmapsto \nabla_X Y$ that satisfies the following conditions:*

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
2. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ *(Leibniz rule)*

   for all $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{F}(M)$

Definition 1.1.4. *Let $\alpha : I \longrightarrow M$ be a curve in a manifold M, a vector field along the curve α is a smooth map such that: for every $t \in I$ gives a tangent vector $V(t) \in T_{\alpha(t)}M$. To say that V is smooth means that for any smooth function f on M, the function $t \longrightarrow V(t)f$ is a smooth function on I. Where I is an open interval in $\mathbb{R}$.*

Proposition 1.1.2. *Let M be a Riemannian manifold with connection ∇, and α a curve of M. Then there exists a unique operator that associates to a vector field V along the curve α another vector field $V'(t) = D_\alpha V(t)$ along α, such that:*

1. $D_\alpha(aV + bW) = aD_\alpha V + bD_\alpha W, a, b \in \mathbb{R}$
2. $D_\alpha(fV) = \frac{df}{dt}V + fD_\alpha V, f \in \mathcal{F}(I)$
3. $\frac{d}{dt}\langle V(t), W(t)\rangle = \langle D_\alpha V(t), W(t)\rangle + \langle V(t), D_\alpha W(t)\rangle$
4. *If* $V(t) = Y(\alpha(t))$ *where* $Y \in \mathcal{X}(M)$, *then* $D_\alpha V(t) = \nabla_{\alpha'(t)} Y$

In the special case that $D_\alpha V(t) = 0$, the vector field V along α is called parallel. The following definition is motivated from the notion of a parallel vector field along a curve:

Definition 1.1.5. *A geodesic in a Riemannian manifold M is a curve $\gamma : I \longrightarrow M$ whose vector field γ' is parallel, that is,*

$$\nabla_{\gamma'}\gamma' = 0$$

Remark 1. *A linear connection ∇ on a Riemannian manifold $(M, \langle , \rangle)$ is called compatible with the Riemannian metric if, for any smooth curve $\alpha : I \longrightarrow M$ and for any $t_0, t_1 \in I$, the parallel transport $\tau_{t_0,t_1} : T_{\alpha(t_0)}M \longrightarrow T_{\alpha(t_1)}M$ preserves the scalar product.*

Proposition 1.1.3. *Let $(M, \langle , \rangle)$ be a Riemannian manifold and ∇ a linear connection. The following points are equivalent:*

1. *∇ is compatible with the metric.*
2. *For any $\alpha : I \longrightarrow M$ and any vector fields V, W along α*
$$\frac{d}{dt}\langle V(t), W(t)\rangle = \langle D_\alpha V(t), W(t)\rangle + \langle V(t), D_\alpha W(t)\rangle$$
3. *For any $X, Y, Z \in \mathcal{X}(M)$,*
$$\nabla_X\langle Y, Z\rangle = X\langle Y, Z\rangle - \langle \nabla_X Y, Z\rangle - \langle \nabla_X Z, Y\rangle = 0$$

Levi-Civita connection The following theorem is a fundamental result in Riemannian geometry

Theorem 1.1.1. *Given a Riemannian manifold $(M, \langle , \rangle$, there exists a unique linear connection ∇ (called the Levi-Civita or Riemannian connection) such that:*
(i) $\nabla_X Y - \nabla_Y X - [X, Y] = 0, \forall X, Y \in \mathcal{X}(M)$.
(ii) ∇ is compatible with the metric, this is equivalent to

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle \nabla_X Z, Y\rangle, \forall X, Y, Z \in \mathcal{X}(M)$$

This connection is characterized by the Koszul formula.

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = \quad & X.\langle Y, Z\rangle + Y.\langle X, Z\rangle - Z.\langle X, Y\rangle \\ & + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle. \end{aligned}$$

Proof. To compute $\nabla_X Y$ it is sufficient to compute $\langle \nabla_X Y, Z\rangle$, because $\langle , \rangle$ is nondegenerate. By using over and over the fact that ∇ is compatible with the metric and torsion free, we get

$$\begin{aligned}
\langle \nabla_X Y, Z\rangle &= X.\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle \\
&= X.\langle Y, Z\rangle - \langle Y, \nabla_Z X\rangle - \langle Y, [X, Z]\rangle \\
&= X.\langle Y, Z\rangle - Z.\langle Y, X\rangle + \langle \nabla_Z Y, X\rangle - \langle Y, [X, Z]\rangle \\
&= X.\langle Y, Z\rangle - Z.\langle Y, X\rangle + \langle \nabla_Y Z, X\rangle + \langle [Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\
&= X.\langle Y, Z\rangle - Z.\langle Y, X\rangle + Y.\langle Z, X\rangle - \langle Z, \nabla_Y X\rangle + \langle [Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\
&= X.\langle Y, Z\rangle - Z.\langle Y, X\rangle + Y.\langle Z, X\rangle - \langle Z, \nabla_X Y\rangle + \langle Z, [X, Y]\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
2\langle \nabla_X Y, Z\rangle = \; & X.\langle Y, Z\rangle + Y.\langle X, Z\rangle - Z.\langle X, Y\rangle \\
& + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle.
\end{aligned} \tag{1.1}$$

This formula gives the uniqueness and can be used to define ∇. $\square$

We will give the notion of curvature, the Riemann curvature tensor is one of the basic invariants of a Riemannian manifold. In fact, Riemann introduced the notion of the sectional curvature in geometric manner as an extension of the Gaussian curvature for surfaces to arbitrary Riemannian manifolds. His definition was not a practical. It took several years to reach a formulation that is easy to use to prove theorems.

Definition 1.1.6. *Let M be a Riemannian manifold, with Levi-Civita connection ∇. The Riemann curvature tensor is the function*

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

given by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z. \tag{1.2}$$

Theorem 1.1.2. *Let M be a Riemannian manifold and R a Riemann curvature tensor, then R satisfying the following:*

1. *R is a tensor field of type $(3, 1)$.*

2. *R is the unique tensor field satisfying for any variation $(s, t) \longrightarrow \Gamma(s, t) \in M$ and any vector field along Γ*

$$D_S D_T Y - D_T D_S Y = -R(S, T)Y. \tag{1.3}$$

Proof. 1. We will show that R given by 1.2 is a tensor field, i.e., it is $C^\infty(M)$ 3-linear. Let $f \in C^\infty(M)$. we have

$$\begin{aligned}
-R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z\\
&= f\nabla_X\nabla_Y Z - \nabla_Y f\nabla_X Z - \nabla_{f[X,Y]-Y(f)X}Z\\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z\\
&= -fR(X,Y)Z.
\end{aligned}$$

Since $R(X,Y)Z = -R(Y,X)Z$, then we have $R(X,fY)Z = fR(X,Y)Z$. On the other hand, we have

$$\begin{aligned}
-R(X,Y)fZ &= \nabla_X\nabla_Y fZ - \nabla_Y\nabla_X fZ - \nabla_{[X,Y]}fZ\\
&= \nabla_X(f\nabla_Y Z + Y(f)Z) - \nabla_Y(f\nabla_X Z + X(f)Z) - f\nabla_{[X,Y]}Z - [X,Y](f)Z\\
&= f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z\\
&\quad -X(f)\nabla_Y Z - Y(X(f))Z - f\nabla_{[X,Y]}Z - [X,Y](f)Z\\
&= -fR(X,Y)Z,
\end{aligned}$$

2. Put $\Gamma(s,t) = (x_1(s,t),...,x_n(s,t))$, where $(x_1,...,x_n)$ a coordinates system and let $Y = \sum_{i=1}^n Y_i(s,t)\partial_i$. We have

$$T(s,t) = \sum_{i=1}^n \frac{\partial x_i}{\partial t}\partial_i \quad \text{and} \quad S(s,t) = \sum_{i=1}^n \frac{\partial x_i}{\partial s}\partial_i$$

$$D_T Y = \sum_{i=1}^n \frac{\partial Y_i}{\partial t}\partial_i + \sum_{i,j=1}^n Y_i\frac{\partial x_j}{\partial t}\nabla_{\partial_j}\partial_i$$

$$D_S D_T Y = \sum_{i=1}^n \frac{\partial^2 Y_i}{\partial s\partial t}\partial_i + \sum_{i,j=1}^n \left(\frac{\partial Y_i}{\partial s}\frac{\partial x_j}{\partial t} + \frac{\partial Y_i}{\partial t}\frac{\partial x_j}{\partial s} + \frac{\partial^2 x_j}{\partial t\partial s}\right)\nabla_{\partial_i}\partial_j + \sum_{i,j,k=1}^n Y_i\frac{\partial x_j}{\partial t}\frac{\partial x_k}{\partial s}\nabla_{\partial_k}\nabla_{\partial_j}\partial_i$$

In the same way we get $D_T D_S Y$. Then we get

$$\begin{aligned}
(D_S D_T - D_T D_S)Y &= \textstyle\sum_{i,j,k=1}^n Y_i\frac{\partial x_j}{\partial t}\frac{\partial x_k}{\partial s}(\nabla_{\partial_k}\nabla_{\partial_j}\partial_i - \nabla_{\partial_j}\nabla_{\partial_k}\partial_i)\\
&= -\textstyle\sum_{i,j,k=1}^n Y_i\frac{\partial x_j}{\partial t}\frac{\partial x_k}{\partial s}R(\partial_j,\partial_k)\partial_i\\
&= -R(S,T)Y
\end{aligned}$$

3. Now we show the uniqueness of R, Let R' be a $(3,1)$-tensor field satisfying 1.3. Let $p \in M$ and $\psi = (x_1,\dots,x_n)$ a coordinates system around p satisfying $\psi(p) = 0$. For $i,j \in \{1,...,n\}$ fixed, we consider the variation Γ given by

$$\Gamma(s,t) = \psi^{-1}(0,\dots,s,\dots,t,\dots,0)$$

, s at the i-place and t at the j-place. We have $S = \partial_i$ and $T = \partial_j$ and let $Y = \partial_k$. Since R' satisfies 1.3, we get

$$R'(\partial_i, \partial_j)\partial_k = -(D_S D_T - D_T D_S)Y = R(\partial_i, \partial_j)\partial_k$$

thus $R' = R$.

□

A simpler real-valued function that completely determines R is the sectional curvature.

Definition 1.1.7. *Let $(M, \langle , \rangle)$ be a Riemannian manifold. For any $p \in M$ Let V be a two-dimensional subspace of T_pM and let $u, v \in V$ be two linearly independent vectors. Then the number*

$$Q(u, v) = \frac{\langle R(u, v)u, v\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}$$

does not depend on the choice of the vectors u, v. It is called the sectional curvature of V at p.

Theorem 1.1.3. *The curvature tensor at a point p is uniquely determined by the sectional curvatures of all the two-dimensional subspaces V of the tangent space T_pM.*

A Riemannian manifold is said to have constant sectional curvature (positive or negative) if $Q(V)$ is a constant (positive or negative) for all planes V in T_pM and for all points $p \in M$. If the sectional curvature is zero at every point, then the Riemannian manifold is said to be flat.

Definition 1.1.8. *Let $(M, \langle , \rangle)$ be a Riemannian metric and R its curvature tensor. The Ricci curvature $Ric(X, Y)$ of M is the trace of the map $Z \longrightarrow R(X, Z)Y$*

Let $(e_1, ..., e_n)$ be an orthonormal basis of T_pM, we have for any $u, v \in T_pM$,

$$\begin{aligned}
\mathrm{ric}(u, v) &= \mathrm{tr}(x \longmapsto R(u, x)v) \\
&= \textstyle\sum_{i=1}^n \langle R(u, e_i)v, e_i\rangle \\
&= \textstyle\sum_{i=1}^n \langle R(v, e_i)u, e_i\rangle \\
&= \mathrm{ric}(v, u)
\end{aligned}$$

We can see the Ricci curvature as an endomorphism $Ric_p : T_pM \longrightarrow T_pM$ such that

$$\mathrm{ric}(u, v) = \langle Ric_p(u), v\rangle = \langle Ric(v), u\rangle, \forall u, v \in T_pM$$

Thus

$$Ric_p(u) = \sum_{i=1}^n R(u, e_i)e_i.$$

We call *Ric* Ricci operator. It is a symmetric field of endomorphism and hence has real eigenvalues in each point $p \in M$ say $\lambda_p^1 \leq ... \leq \lambda_p^n$. The metric has positive (resp. strictly positive) Ricci curvature if, for any $i = 1, ..., n$, $\lambda_p^i \geq 0$ (resp. $\lambda_p^i > 0$). In analogue way, we can define metric of negative Ricci curvature and strictly negative curvature. The metric is called Einstein if ric $= \lambda\langle, \rangle$ where λ is a constant.

Definition 1.1.9. *The scalar curvature of $(M, \langle, \rangle)$ is the C^∞ function $s : M \longrightarrow \mathbb{R}$ given by*

$$s(p) = \mathrm{tr}(Ric_p) = \sum_{i=1}^{n} \lambda_p^i = \sum_{i=1}^{n} \mathrm{ric}(e_i, e_i)$$

1.2 Lie groups

Definition 1.2.1. *A Lie group is a group G, equipped with a manifold structure such that the group operations*
(i) The map $p : G \times G \to G$ defined by $p(g, h) = gh$ is smooth when $G \times G$ is endowed with the product manifold structure.
(ii) The map $inv : G \to G$ defined by $inv(g) = g^{-1}$ is smooth.
A morphism of Lie groups G, G' is a morphism of groups $\varphi : G \longrightarrow G'$ that is smooth.

Example 2. *Any discrete group G is a Lie group of dimension zero. In particular $\mathbb{Z}$ or more generally $\mathbb{Z}^n$, is a Lie group. It is a closed subgroup of $\mathbb{R}^n$.*

Example 3. *The multiplicative group $\mathbb{R}^*$ is a Lie group. It is not connected. $\mathbb{R}^*_+$ of positive real numbers is also a Lie group. Similarly, $\mathbb{C}^*$ is a (2 dimensional) Lie group which is connected. It is a complex Lie group.*

Example 4. *The unit circle $\mathbb{S}^1$ is a Lie group. There are two ways to see this. One is by considering $\mathbb{S}^1$ in $\mathbb{C}^*$ with multiplication induced from $\mathbb{C}^*$ The other is by using the identification $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The set $\mathbb{Z}$ of integers is a normal subgroup of $\mathbb{R}$, and so $\mathbb{R}/\mathbb{Z}$ is a group, and since it is discrete, $\mathbb{R}/\mathbb{Z}$ is also a manifold. The smooth addition of $\mathbb{R}$ induces a smooth addition in $\mathbb{R}/\mathbb{Z}$.*

Example 5. *The product $G \times H$ of two Lie groups is itself a Lie group with the product manifold structure, and multiplication $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$, as example for this The n-torus $T^n = \mathbb{S}^1 \times ... \times \mathbb{S}^1$ (n times) is a Lie group of dimension n.*

Example 6. *$GL_n(\mathbb{R})$, is a (dense) open subset of $M_n(\mathbb{R})$ and thus has a manifold structure in which multiplication is a polynomial function of the coordinates. Moreover, inversion is a rational function of the coordinates with a non vanishing denominator. Hence $GL_n(\mathbb{R})$ is a real Lie group of dimension n^2.*

The following examples of Lie groups are obtained as closed subgroups of the general linear group, so we need the following definitions.

Definition 1.2.2. *A Lie subgroup of a Lie group G is a Lie group H that is an abstract subgroup and an immersed submanifold of G.*

We have the following theorem :

Theorem 1.2.1 (E. Cartan's theorem.)**.** *Let H be a closed subgroup of a Lie group G. Then H is an embedded submanifold, and hence is a Lie subgroup.*

We can now give more examples of Lie groups that are defined by using functions on $M_n(\mathbb{R})$ as the determinant, transpose and complex conjugate, hence are Lie groups by the previous theorem.

Example 7. *(1) The special linear group is $SL(n, \mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}$.*

(2) The orthogonal group is the group $O(n) = \{A \in GL_n(\mathbb{R}) : A^t A = I_n\}$. The condition $A^t A = I_n$ is equivalent to $A^{-1} = A^t$ and so $O(n) = F^{-1}(I_n)$, where $F : GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$ with $F(A) = A^t A$.

(3) The special. orthogonal group is the group $SO(n) = \{A \in O(n) : \det(A) = 1\}$.

Let x be an element of a Lie group G. We define the maps

$$L_x : G \longrightarrow G, L_x(g) = xg \text{ (left translation).}$$
$$R_x : G \longrightarrow G, R_x(g) = gx \text{ (right translation)}$$

These maps are smooth, in fact they are diffeomorphisms since, the inverse of L_x is $L_{x^{-1}}$. We have $(dL_{x^{-1}})_x : T_x G \longrightarrow T_e G$ is isomorphism of vector spaces, then we have the following:

Proposition 1.2.1. *Any Lie group G is parallelizable, i.e. $TG \cong G \times T_e G$*

Definition 1.2.3. *A vector field X on G is left invariant if for all $g \in G$, we have:*

$$X_{gx} = T_x L_g(X_x)$$

The set of left-invariant vector fields on G is denoted $\mathcal{X}^l(G)$, it is a Lie subalgebra of $\mathcal{X}(G)$ for the bracket of vector fields, since it is closed under the bracket on Vector fields.
In fact, $\mathcal{X}^l(G)$ is a real vector space of finite dimension equal to the dimension of G and this result comes from that the map $\phi : \mathcal{X}^l(G) \longrightarrow T_eG$ given by $X \longmapsto X_e$ is an isomorphism of vector spaces. The isomorphism ϕ allows to transport the Lie algebra structure of $\mathcal{X}^l(G)$ on T_eG as follows: if we denote for all $u \in T_eG, v^l := \phi^{-1}(v) \in \mathcal{X}^l(G)$ we get a structure of Lie algebra on T_eG given by the bracket:

$$[u, v] := [u^l, v^l]_e.$$

Then we call Lie algebra of G and we denote by $\mathfrak{g} = T_eG$.

Proposition 1.2.2. *If $\phi : G \longrightarrow H$ is a homomorphism of Lie groups, then the map $d_e\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a homomorphism of Lie algebras,*

These are Lie's results:

Theorem 1.2.2 (An). *(i) For any Lie algebra $\mathfrak{g}$ there is a Lie group G (not necessarily unique) whose Lie algebra is $\mathfrak{g}$.*

(ii) Let G be a Lie group with Lie algebra $\mathfrak{g}$. If H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Conversely, for each Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a unique connected Lie subgroup H of G which has $\mathfrak{h}$ as its Lie algebra.

(iii) Let G_1, G_2 be Lie groups with corresponding Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$. Then if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic as Lie algebras, then G_1 and G_2 are locally isomorphic. If the Lie groups G_1, G_2 are simply connected (i.e. their fundamental groups are trivial), then G_1 is isomorphic to G_2.

1.3 Action of Lie groups on manifolds and representations

Definition 1.3.1. *Let G be a group and M a set. Then G is said to act on M (on the left) if there is a map $\theta : G \times M \to M$ such that:*
(i) If e is the identity element of G then

$$\theta(e, x) = x \quad for \quad all \quad x \in M$$

(ii) If g_1,$g_2 \in G$, then

$$\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x) \quad for \quad all \quad x \in M$$

$\theta(g, x)$ denoted by $g.x$

A right action is defined analogously as map $\theta : M \times G \longrightarrow G$ with $\theta(x, g) = x.g$.
Now suppose G is a Lie group and M is a manifold. An action of G on M is said to be continuous if the map θ is continuous, and it is smooth if the map θ is smooth.
The conditions (i) and (ii) for a left action give

$$\begin{aligned} \theta(g_1, .) \circ \theta(g_2, .) &= \theta(g_1 g_2, .) \\ \theta(e, .) &= Id_M \end{aligned}$$

Thus, for a continuous action each $\theta(g, .) : M \longrightarrow M$ is a homeomorphism because it has an inverse $\theta(g^{-1}, .)$ continuous. If the action is smooth, then each $\theta(g, .)$ is a diffeomorphism.
• For any $x \in M$, the orbit of x under the action is the sets

$$Gx = \{g.x : g \in G\},$$

the set of all images of x under the action by elements of G.

• The action is **transitive** if for any two points x,$y \in M$, there is a group element g such that $g.x = y$, or equivalently if the orbit of any point is all of M.

• Given $x \in M$, the **isotropy group** of x, denoted by G_x, is the set of elements $g \in G$ that fix x:

$$G_x = \{g \in G : g.x = x\}.$$

• The action is said to be **free** if the only element of G that fixes any element of M is the identity: $g.x = x$ for some $x \in X$ implies $g = e$. This is equivalent to the requirement that $G_x = \{e\}$ for every $x \in M$.

Example 8. *The natural action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$ is the left action given by matrix multiplication: $(A, x) \mapsto Ax$, considering $x \in \mathbb{R}^n$ as a column matrix. This an action because $I_n x = x$ and matrix multiplication is associative:$(AB)x = A(Bx)$. Because any nonzero vector can be taken to any other by some linear transformation, there are exactly two orbits: $\{0\}$ and $\mathbb{R}^n \diagdown \{0\}$.*

Example 9. *The restriction of the natural action to $O(n) \times \mathbb{R}^n \to \mathbb{R}^n$ defines a left action of $O(n)$ on $\mathbb{R}^n$. Any nonzero vector of length R can be taken to any other by an orthogonal* *matrix. (If v and v' are such vectors, complete $v/|v|$ and $v'/|v'|$ to orthonormal bases and let A and A' be the orthogonal matrices whose columns are these orthonormal bases; then $A'A^{-1}$ takes v to v'). In this case, the orbits are the origin and the spheres centered at the origin.*

Example 10. *The restriction of the natural action to $O(n) \times \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$, we obtain a transitive action of $O(n)$ on $\mathbb{S}^{n-1}$.*

Example 11. *The natural action of $O(n)$ restricts to an action of $SO(n)$ on $\mathbb{S}^{n-1}$. For $SO(1)$, this is trivial because $SO(1) = \{1\}$. But For $n > 1$, $SO(n)$ acts transitively on $\mathbb{S}^{n-1}$. Since $O(n)$ acts transitively, there is a matrix $A \in O(n)$ taking e_1 to v. Either $detA = 1$, in which case $A \in SO(n)$, or $detA = -1$, in which case the matrix obtained by multiplying the last column of A by -1 is in $SO(n)$ and still takes e_1 to v.*

1.3.1 Representations theory

There are different reasons to look for the representations. For example, a representation is a useful tool for understanding the group and its possible invariants. Since the Lie groups are often the symmetry groups of spaces of functions, finding the ways in which a group can act helps to understand these spaces.

Definition 1.3.2. *Let G be a Lie group. A (finite-dimensional) representation of G is a Lie homomorphism $\rho : G \longrightarrow Aut(V)$, where V is a (finite-dimensional) vector space. The dimension of the representation is the dimension of the vector space V. Where $Aut(V) = GL(V)$*

Any representation ρ defines a smooth action of G on V:

$$g.v = \rho(g)v, for \quad g \in G, v \in V$$

Definition 1.3.3. *An action of G on a finite-dimensional vector space V is said to be linear if for each $g \in G$ the map $v \longmapsto g.v$ is linear.*

Remark 2. *We have for any representation $\rho : G \longrightarrow GL(V)$ the action $g.v = \rho(g)v, for \quad g \in G, v \in V$ is linear. And the image of ρ is a Lie subgroup og $GL(V)$, if ρ is injective then the representation is called **faithful representation***

Proposition 1.3.1. *Let G be a Lie group and let V be a finite-dimensional vector space, then a smooth action of G on V is linear if and only if it has a form for some representation ρ of G.*

Example 12. *Let $G = \mathbb{R}^n$ and $V = \mathbb{R}^{n+1}$, put $\rho : \mathbb{R}^n \longrightarrow GL(n+1, \mathbb{R})$ the map sends $x \in \mathbb{R}^n$ to the matrix $\rho(x) = \begin{pmatrix} I_n & x \\ 0 & 1 \end{pmatrix}$, we have ρ is a faithful representation of the Lie group $\mathbb{R}^n$.*

Example 13. *If G is any Lie subgroup of $GL(n, \mathbb{R})$, the inclusion map $G \hookrightarrow GL(n, \mathbb{R})$ is a faithful representation.*

Example 14. *Let $G = \mathbb{R}^n$ and $V = \mathbb{C}^n$, put $\rho : \mathbb{R}^n \longrightarrow GL(n, \mathbb{C})$ for $x \in \mathbb{R}^n$, $\rho(x)$ is the diagonal matrix with diagonal entries $(e^{2\pi i x_1}, ..., e^{2\pi i x_n})$. This action is not faithful, since its kernel is $\mathbb{Z}^n$.*

Example 15. ***The adjoint representation***
Let G be a Lie group and $x \in G$. Then the map $C_x : G \longrightarrow G$ sending each g to :xgx^{-1} is a homomorphism and, because $C_x = R_{x^{-1}} \circ L_x$ is a diffeomorphism, it is called an inner automorphism of G. We let $Ad(g) = (d_e C_g) : \mathfrak{g} \longrightarrow \mathfrak{g}$. This is a homomorphism since $C_{xy} = C_x \circ C_y$ implies that $Ad(xy) = Ad_x \circ Ad_y$ (We take differentials). And $Ad(x)$ is invertible with inverse $Ad(x^{-1})$. It is also smooth (see [Lee]). Then $Ad : G \longrightarrow GL(\mathfrak{g})$ is a representation, called the adjoint representation.

Remark 3. *If ρ is any representation of G, then $d_e\rho : \mathfrak{g} \longrightarrow GL(\mathfrak{g})$ is a representation of $\mathfrak{g}$, when $\mathfrak{g}$ is the Lie algebra of G. As an example $ad(u) = d_e Ad(u)$ is called the adjoint representation of $\mathfrak{g}$.*

Proposition 1.3.2. *The correspondence $ad : \mathfrak{g} \longrightarrow End(\mathfrak{g}), u \longmapsto ad_u$ is called the adjoint representation of Lie algebra $\mathfrak{g}$. Moreover, if $u, v \in \mathfrak{g}$, then:*

$$ad_u(v) = [u, v]$$

Remark 4. *$\mathfrak{g}$ is unimodular if and only if $\operatorname{tr}(ad_u) = 0, \forall u \in \mathfrak{g}$.*

Definition 1.3.4. *A continuous action is said to be* ***proper*** *if the map:*

$$\begin{aligned} G \times M &\longrightarrow M \times M \\ (g, x) &\longmapsto (g.x, x) \end{aligned}$$

is a proper map.Where proper map is defined as a map between topological spaces such that, its preimage of a compact subset is compact itself.

Proposition 1.3.3. *Let M be a manifold, and let G be a Lie group acting continuously on M. Then, the action is proper if and only if following holds: If (p_i) is a convergent sequence in M and (g_i) is sequence in G such that $(g_i.p_i)$ converge in M , then a subsequence of (g_i) converge in G.*

Corollary 1.3.1. *Any continuous action by a compact Lie group on a manifold is proper.*

1.4 Orbits and homogeneous spaces

Let a Lie group G acts on a manifold M. We can define an equivalence relation on M by setting $p \sim q$ if there exists $g \in G$ such that $g.p = q$. The set of orbits is denoted by M/G; with the quotient topology it is called the orbit space of the action. It is important to determine conditions such that an orbit space is a smooth manifold.

Theorem 1.4.1 (Quotient Manifold Theorem see [15])**.** *Suppose G is a Lie group acting smoothly,freely and properly on a smooth manifold M.Then the orbit space M/G is a topological manifold of dimension equal to $dim(M) - dim(G)$, and has a unique smooth structure with the property that the quotient map $\pi : M \longrightarrow M/G$ is a smooth submersion.*

Let G be a Lie group and H a closed subgroup of G, it is possible to make a smooth manifold on the set $G/H = \{gH : g \in G\}$. Furthermore, We will see that the group G acts in a natural way on G/H , and this action such that any two points in G/H can be joined by the action of G, i.e., the action is transitive. This manifold with this transitive action will be called a homogeneous space, and it includes a large variety of manifolds with special importance in mathematics and physics.

Consider the coset space G/H. Let $\pi : G \longrightarrow G/H$ denote the projection that sends each $g \in G$ to the coset gH.

Theorem 1.4.2 (see [15])**.** *Let G be a Lie group, and H a closed subgroup of G. Then there is a unique way to make G/H a manifold so that the projection $\pi : G \longrightarrow G/H$ is a submersion.*

Theorem 1.4.3 (see [15])**.** *Let $G \times M \longrightarrow M$ be a transitive action of a Lie group G on a manifold M, and let $H = G_x$ be the isotropy subgroup of a point $x \in M$. Then:*

1. *The subgroup H is a closed subgroup of G.*

2. *The map $\gamma : G/H \longrightarrow M$ given by $\gamma(gH) = g.x$ is a diffeomorphism.(the orbit $G.x$ is diffeomorphic to G/H).*

3. *The dimension of G/H is* $\dim G - \dim H$*.*

Definition 1.4.1. *A homogeneous space is a manifold M with a transitive action of a Lie group G. Equivalently, it is a manifold of the form G/H , where G is a Lie group and H a closed subgroup of G.*

Example 16. *Any Lie group is a homogeneous space, since there are two representations: $G = G \times G/G = G/\{e\}$. For the first representation of G as a homogeneous space, $G \times G$ acts on G by left and right translations, and the isotropy subgroup is G diagonally embedded in $G \times G$.*

Definition 1.4.2. *Let M be a Riemannian manifold, we call M A Riemannian homogeneous space on which its isometry group $I(M)$ acts transitively.*

Remark 5. *The isometry group of a Riemannian manifold is a Lie group (this is theorem of Myers-Steenrod).*

Example 17. *The group $O(n+1)$ acts on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ when $n \geq 1$. This action is transitive as in example 10. In Example 11, $SO(n+1)$ acts transitively on $\mathbb{S}^n$. Then for $n \geq 1$, $\mathbb{S}^n$ is a homogeneous space. Thus $\mathbb{S}^n$ is diffeomorphic to the quotient manifold $O(n+1)/O(n)$ and also it is diffeomorphic to $SO(n+1)/SO(n)$*

Example 18. *The Euclidean group $E(n) = \{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} : A \in O(n), b \in \mathbb{R}^n\}$ acts transitively on $\mathbb{R}^n$ by the action $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \bullet x = Ax + b$. Thus $\mathbb{R}^n$ is a homogeneous $E(n)$- space.*

Example 19. *$SL(2,\mathbb{R})$ acts transitively by Möbius transformations ($\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}$) on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Imz > 0\}$, then $\mathbb{H}$ is homogeneous space and we have the isotropy group of $i \in \mathbb{H}$ is exactly $SO(2)$. Thus $\mathbb{H} \approx SL(2,\mathbb{R})/SO(2)$.*

Example 20 (Grassmanna Manifolds)**.** *The set of all k-dimensional (vector) subspaces $V \subset \mathbb{R}^n$ is called the Grassmann manifold of k-planes in $\mathbb{R}^n$ and denoted by $G_{k,n}(\mathbb{R})$. The group $O(n)$ acts naturally on $G_{k,n}(\mathbb{R})$ by matrix multiplication, for V k-plane in $\mathbb{R}^n$ gives $A.V = W$. This action is transitive: Let V be a k-plane in $\mathbb{R}^n$ spanned by the first k vectors of the canonical basis $\{e_1, ..., e_n\}$ of $\mathbb{R}^n$. Let W be a k-plane in $\mathbb{R}^n$ spanned by the first k vectors of an orthonormal* *basis* $\{e'_1, ..., e'_n\}$ *of* $\mathbb{R}^n$ *if* A *is the matrix that corresponds to the linear map that sends each* e_i *to* e'_i, *then* $A \in O(n)$ *and* $AV = W$ *The isotropy subgroup of the subspace* V *is:*

$$\{\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, A_1 \in O(k) \quad and \quad A_2 \in O(n-k)\}$$

Thus $G_{k,n}(\mathbb{R})$ *is diffeomorphic to* $O(n)/(O(k) \times O(n-k)$.

Example 21 (Flag Manifolds)**.** *Let* E *be a real vector space of dimension* $n > 1$, *and let* $K = (k_1, ..., k_m)$, *where* $(k_i)_{i=1}^m$ *satisfying* $0 < k_1 < ... < k_m < n$. *A flag in* E *of type* K *is a nested sequence of linear subspaces* $V_1 \subset V_2 \subset ... \subset V_m \subset E$, *with* $dim V_i = k_i$ *for each* i. *The set of flag of type* K *in* E *is denoted by* $F_K(E)$, *we have* $GL(E)$ *acts transitively on* $F_K(E)$.

1.5 Invariant Riemannian metrics

Let G be a Lie group, we have G is a smooth manifold and it is a group, it is usual to use Riemannian metrics of G with its group structure. These metrics have the property that the left translations $L_a : G \longrightarrow G$ are isometries for all $a \in G$. More precisely, we have:

Definition 1.5.1. *A Riemannian metric on a Lie group* G *is called left invariant if for all* $a \in G$ $L_a^* g = g$ *e.i. For all* $x, a \in G$,

$$g_{ax}(T_x L_a(u), T_x L_a(v)) = g_x(u, v), u, v \in T_x G$$

Similarly, a Riemannian metric is right invariant if each $R_a : G \longrightarrow G$ *is an isometry.*

Denote by $\mathfrak{g}$ the Lie algebra of G, $\mathcal{M}^l(G)$ the set of left invariant metrics on G and $\mathcal{M}(\mathfrak{g})$ the set of scalar products on $\mathfrak{g}$.

Proposition 1.5.1. *There is a one-to-one correspondence between left invariant metrics on a Lie group* G, *and scalar products on its Lie algebra* $\mathfrak{g}$ *i.e the map* $\Phi : \mathcal{M}^l(G) \longrightarrow \mathcal{M}(\mathfrak{g})$ *is a bijection.*

Proposition 1.5.2. *Let* X, Y *be two left invariant vector fields on* G *and* g *be a left invariant metric on* G. *Then the function* $G \longrightarrow \mathbb{R}$, $x \longmapsto g_x(X_x, Y_x)$ *is constant equal to* $g_e(X_e, Y_e)$.

Proof. Let $x \in G$, since X and Y are left invariant then $X_x = T_e L_x(X_e)$ and $Y_x = T_e L_x(Y_e)$. And we have g is left invariant, we get that:

$$g_x(X_x, Y_x) = g_x(T_e L_x(X_e), T_e L_x(Y_e)) = L^* g_e(X_e, Y_e) = g_e(Xe, Y_e).$$

□

Definition 1.5.2. *We call Riemannian Lie group any pair* (G, g) *where* G *is a Lie group and* g *is a left invariant metric on* G.

Definition 1.5.3. *A metric* g *on a Lie group* G *is said to be bi-invariant when it is both right and left invariant.*

Proposition 1.5.3. *There is a one-to-one correspondence between bi- invariant metrics on* G *and* Ad*-invariant scalar products on* $\mathfrak{g}$, *that is* $\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = \langle X, Y\rangle$ *for all* $g \in G, X, Y \in \mathfrak{g}$.

Now, regarding the existence of bi-invariant metrics, here is a theorem which determines the class of Lie groups admitting such metrics:

Theorem 1.5.1. *(i) A connected Lie group carries a bi-invariant Riemannian metric if and only if it is isomorphic to the cartesian product of a compact Lie group and an abelian Lie group.*
(ii) If the Lie algebra of a compact Lie group G *is simple, then* G *admits a bi-invariant Riemannian metric which is unique up to positive multiplicative constant.*

In general, the existence of a bi-invariant metric on a Lie group is a precise problem and there are examples where the answer is not affirmative. However, if (G, g) is a Riemannian Lie group, there is a way to measure the obstruction for g to be bi-invariant. For this, we pose:

$$I(g) = \{x \in G : \mathrm{Ad}_x \quad is \quad an \quad isometry \quad of(\mathfrak{g}, g_e = \langle , \rangle)\}$$

We have $I(g)$ is a subgroup of G and that g is bi-invariant if and only if $I(g) = G$. On the other hand, the group $I(g)$ is closed in G and it therefore has a Lie group structure. We denote in the following $K(g)$ the Lie algebra of $I(g)$.

Proposition 1.5.4. *We have* $K(g) = \{u \in \mathfrak{g} : \mathrm{ad}_u + \mathrm{ad}_u^* = 0\}$ *where* ad_u^* *adjoint of* ad_u.

Proof. Let $u \in K(g)$, we have $\forall t \in \mathbb{R}$, $\exp(tu) \in I(g)$. In other words, we have for all $v, w \in \mathfrak{g}$

$$\langle \mathrm{Ad}_{\exp(tu)} v, \mathrm{Ad}_{\exp(tu)} w\rangle = \langle v, w\rangle$$

By deriving at t = 0 we obtain that:

$$\langle \mathrm{ad}_u(v), w\rangle + \langle v, \mathrm{ad}_u(w)\rangle = 0.$$

Thus $\mathrm{ad}_u + \mathrm{ad}_u^* = 0$.

Conversely, Let $u \in \mathfrak{g}$ such that $\mathrm{ad}_u + \mathrm{ad}_u^* = 0$ we need to prove that $u \in K(g)$, we define for all $t \in \mathbb{R}$

$$f(t) = \langle \mathrm{Ad}_{\exp(tu)} v, \mathrm{Ad}_{\exp(tu)} w \rangle$$

By calculation the derivative of f we have

$$\begin{aligned} f'(t) =& \tfrac{d}{dt} \langle \mathrm{Ad}_{\exp(tu)} v, \mathrm{Ad}_{\exp(tu)} w \rangle \\ =& \tfrac{d}{ds}|_{s=0} \langle \mathrm{Ad}_{\exp((s+t)u)} v, \mathrm{Ad}_{\exp((s+t)u)} w \rangle \\ =& \tfrac{d}{ds}|_{s=0} \langle \mathrm{Ad}_{\exp(su)} \circ \mathrm{Ad}_{\exp(tu)} v, d_{\exp(su)} \circ \mathrm{Ad}_{\exp(tu)} w \rangle \\ =& \langle \mathrm{ad}_u \circ \mathrm{Ad}_{\exp(tu)} v, \mathrm{Ad}_{\exp(tu)} w \rangle + \langle \mathrm{Ad}_{\exp(tu)} v, \mathrm{ad}_u \circ \mathrm{Ad}_{\exp(tu)} w \rangle \\ =& 0 \end{aligned}$$

Thus f is constant, hence $f(t) = f(0)$. Then $\mathrm{Ad}_{exp(tu)}$ is an isometry of $(\mathfrak{g}, g_e = \langle , \rangle)$, this give $\exp(tu) \in I(g)$ for all $t \in \mathbb{R}$ then $u \in K(g)$. □

Corollary 1.5.1. *If the metric g is bi-invariant then ad_u is anti-adjoint with respect to $g_e = \langle , \rangle$ for all $u \in \mathfrak{g}$.*

1.6 Connections on a Riemannian Lie group

Let (G, g) be a Riemannian Lie group and let $\mathfrak{g}$ be the Lie algebra of the group G and $g_e = \langle , \rangle$. Denote by ∇^G the Levi-Civita connection of (G, g), R^G the curvature tensor, Q^G sectional curvature, Ric^G the Ricci tensor of (G, g), ric^G the Ricci curvature of (G, g) and s^G the scalar curvature of (G, g).

Definition 1.6.1. *The Levi-Civita product on Lie algebra $(\mathfrak{g}, \langle , \rangle)$ is the bilinear application $A : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(u, v) \longmapsto A_u v$ given by the formula:*

$$2\langle A_u v, w \rangle = \langle [u, v], w \rangle + \langle [w, u], v \rangle + \langle [w, v], u \rangle$$

Proposition 1.6.1. *Let g be a left invariant metric on a Lie group G. If $\{e_1, ..., e_n\}$ is an orthonormal basis of $(\mathfrak{g}, g_e)$, then the family $\{e_1^l, ..., e_n^l\}$ defines an orthonormal coordinate system of the Lie group (G, g).*

Proposition 1.6.2. *For all $u, v \in \mathfrak{g}$, the vector field $\nabla^G_{u^l} v^l$ is left invariant and we have:*

$$\nabla^G_{u^l} v^l = (A_u v)^l$$

Proof. Let $X = u^l$ and $Y = v^l$ with $u, v \in \mathfrak{g}$. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of $(\mathfrak{g}, g_e)$, then the family $\{e_1^l, ..., e_n^l\}$ defines an orthonormal coordinate system of (G, g). Then

$$\nabla^G_{u^l} v^l = \sum_{i=1}^{n} g(\nabla^G_{u^l} v^l, e_i^l) e_i^l$$

For all $1 \leq i \leq n$, by the Koszul formula 1.1 of the Levi-Civita connection we have

$$\begin{aligned} 2g(\nabla^G_{u^l} v^l, e_i^l) = \quad & u^l.g(v^l, e_i^l) + v^l.g(u^l, e_i^l) - e_i^l.g(u^l, v^l) \\ & + g([u^l, v^l], e_i^l) + g([e_i^l, u^l], v^l) + g([e_i^l, v^l], u^l). \end{aligned}$$

By the proposition 1.5.2 we have $g(v^l, e_i^l)$, $g(u^l, e_i^l)$ and $g(u^l, v^l)$ are constants then $u^l.g(v^l, e_i^l) = v^l.g(u^l, e_i^l) = e_i^l.g(u^l, v^l) = 0$. Thus

$$2g(\nabla^G_{u^l} v^l, e_i^l) = 2\langle A_u v, e_i \rangle$$

Then

$$\nabla^G_{u^l} v^l = \sum_{i=1}^{n} g(\nabla^G_{u^l} v^l, e_i^l) e_i^l = \left(\sum_{i=1}^{n} \langle A_u v, e_i \rangle e_i \right)^l = (A_u v)^l$$

$\square$

Let R^G the curvature tensor of (G, g) then for all $X, Y, Z \in \mathcal{X}(G)$ we have:

$$R^G(X, Y)Z = \nabla^G_{[X,Y]} - [\nabla^G_X, \nabla^G_Y]Z$$

Definition 1.6.2. *We call curvature of the Lie algebra $(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$ the bilinear map $K_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow End(\mathfrak{g})$ given by the formula:*

$$K_{\mathfrak{g}}(u, v) = A_{[u,v]} - [A_u, A_v]$$

Proposition 1.6.3. *For all $u, v, w \in \mathfrak{g}$, the vector field $R^G(u^l, v^l)w^l$ is left invariant on G and we have:*

$$R^G(u^l, v^l)w^l = (K_{\mathfrak{g}}(u, v)w)^l.$$

Proof. Let $u, v, w \in \mathfrak{g}$. we have $[u^l, v^l] = [u, v]^l$ and $\nabla^G_{u^l} v^l = (A_u v)^l$. Then we have:

$$\begin{aligned} R^G(u^l, v^l)w^l &= \nabla^G_{[u^l,v^l]} w^l - \nabla^G_{u^l} \nabla^G_{v^l} w^l + \nabla^G_{v^l} \nabla^G_{u^l} w^l \\ &= \nabla^G_{[u,v]^l} w^l - \nabla^G_{u^l} (A_v w)^l + \nabla^G_{v^l} (A_u w)^l \\ &= (A_{[u,v]} w)^l - (A_u A_v w)^l + (A_v A_u w)^l \\ &= (K_{\mathfrak{g}}(u, v)w)^l \end{aligned}$$

$\square$

Proposition 1.6.4. *Let $x \in G$ and $u, v \in T_xM$, then we have:*

$$Q_x^G(u,v) = Q_e^G(T_xL_x^{-1}(u), T_xL_x^{-1}(v))$$

The following proposition is a consequence of the elementary properties of the curvature tensor R^G.

Proposition 1.6.5. *Let $u, v, w, z \in \mathfrak{g}$ we have the following properties:*

1. $K_{\mathfrak{g}}(u,v) = -K_{\mathfrak{g}}(v,u)$ *anti-symmetry*

2. $K_{\mathfrak{g}}(u,v)w + K_{\mathfrak{g}}(w,u)v + K_{\mathfrak{g}}(v,w)u = 0$

3. $\langle K_{\mathfrak{g}}(u,v)w, z\rangle = \langle K_{\mathfrak{g}}(w,z)u, v\rangle$

4. $\langle K_{\mathfrak{g}}(u,v)w, z\rangle = -\langle K_{\mathfrak{g}}(u,v)z, w\rangle$

Definition 1.6.3. *The Ricci operator of the Lie algebra $(\mathfrak{g}, \langle,\rangle)$ is the linear map $Ric_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{g}$ given by:*

$$\langle Ric_{\mathfrak{g}}(u), v\rangle := \mathrm{tr}(w \longmapsto K_{\mathfrak{g}}(u,w)v) = \sum_{i=1}^{n} \langle K_{\mathfrak{g}}(u,e_i)v, e_i\rangle.$$

where $(e_i)_{i=1}^n$ an orthonormal basis of $(\mathfrak{g}, \langle,\rangle)$.

Proposition 1.6.6. *For all $u, v, w \in \mathfrak{g}$, the vector field $Ric^G(u^l)$ is left invariant on G and we have:*

$$Ric^G(u^l) = (Ric_{\mathfrak{g}}(u))^l$$

Corollary 1.6.1. *The scalar curvature s^G is constant and the function $\mathrm{ric}^G(u^l, v^l)$ is constant for all $u, v \in \mathfrak{g}$.*

1.7 Left invariant metrics on simply connected three dimensional unimodular Lie groups

In this section, we list all the three-dimensional simply connected Lie groups, and for each such G we give all the left invariant Riemannian metrics on G up to automorphis. These results are in reference [7].

They are sixe unimodular simply connected three dimensional unimodular Lie groups:

1. The abelian case is isomorphic to $\mathbb{R}^3$.

2. The nilpotent Lie group Nil known as Heisenberg group whose Lie algebra will be denoted by $\mathfrak{n}$. We have

$$\mathrm{Nil} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\} \quad \text{and} \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{n}$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and where the non-vanishing Lie brackets are $[X_1, X_2] = X_3$.

3. $\mathrm{SU}(2) = \left\{ \begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix}, a^2+b^2+c^2+d^2 = 1 \right\}$ and $\mathfrak{su}(2) = \left\{ \begin{pmatrix} iz & y+ix \\ -y+xi & -zi \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$. The Lie algebra $\mathfrak{su}(2)$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and where the non-vanishing Lie brackets are

$$[X_1, X_2] = X_3, \ [X_2, X_3] = X_1 \quad \text{and} \quad [X_3, X_1] = X_2.$$

4. The universal covering group $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{R})$ whose Lie algebra is $\mathrm{sl}(2, \mathbb{R})$. The Lie algebra $\mathrm{sl}(2, \mathbb{R})$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ X_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad X_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and where the non-vanishing Lie brackets are

$$[X_1, X_2] = -X_3, \ [X_2, X_3] = X_1 \quad \text{and} \quad [X_3, X_1] = X_2.$$

5. The solvable Lie group $\mathrm{Sol} = \left\{ \begin{pmatrix} e^x & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$ whose Lie algebra is $\mathfrak{sol} = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & -x & z \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$. The Lie algebra $\mathfrak{sol}$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and where the non-vanishing Lie brackets are

$$[X_3, X_1] = X_1 \quad \text{and} \quad [X_3, X_2] = -X_2.$$

6. The universal covering group $\widetilde{\mathrm{E}_0}(2)$ of the Lie group

$$\mathrm{E}_0(2) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) & x \\ -\sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix}, \theta, x, y \in \mathbb{R} \right\}.$$

Its Lie algebra is

$$\mathrm{e}_0(2) = \left\{ \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \theta, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathrm{e}_0(2)$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and where the non-vanishing Lie brackets are

$$[X_3, X_1] = X_2 \quad \text{and} \quad [X_3, X_2] = -X_1.$$

In Table 1.1, we collect the informations on these Lie algebras we will use in the chapter three. For each Lie algebra among the five Lie algebras above, we give the set of its homomorphisms and the equivalence classes of Riemannian metrics carried out by this Lie algebra.

These equivalence classes were determined in [7, Theorems 3.3-3.7]. For $\mathfrak{n}$, $\mathfrak{sol}$ and $e_0(2)$ the homomorphisms can be determined easily. For $\mathfrak{su}(2)$ and $\mathrm{sl}(2,\mathbb{R})$ an homomorphism is necessarily an inner automorphism and these were determined in [4].

Lie algebra	Non-vanishing Lie brackets	Homomorphisms	Equivalence classes of Metrics
$\mathfrak{n}$	$[X_1, X_2] = X_3$	$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}$	$\mathrm{Diag}(\lambda, \lambda, 1)$, $\lambda > 0$
$e_0(2)$	$[X_3, X_1] = X_2, [X_3, X_2] = -X_1$	$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix},$ $\begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}, \gamma^2 \neq 1$	$\mathrm{Diag}(1, \mu, \nu)$, $0 < \mu \leq 1, \nu > 0$
$\mathfrak{sol}$	$[X_3, X_1] = X_1, [X_3, X_2] = -X_2$	$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & 0 & a \\ 0 & \beta & b \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & \beta & a \\ \alpha & 0 & b \\ 0 & 0 & -1 \end{pmatrix}, \gamma^2 \neq 1$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$ $\mathrm{Diag}(1, 1, \nu)$ $\nu > 0, \mu > 1$
$\mathrm{sl}(2,\mathbb{R})$	$[X_1, X_2] = -X_3,\ [X_3, X_1] = X_2,$ $[X_2, X_3] = X_1$	$\mathrm{Rot}_{xy}.\mathrm{Boost}_{xz}.\mathrm{Boost}_{yz}$	$\mathrm{Diag}(\lambda, \mu, \nu)$, $0 < \lambda \leq \mu$ and $\nu > 0$
$\mathfrak{su}(2)$	$[X_1, X_2] = X_3,\ [X_3, X_1] = X_2,$ $[X_2, X_3] = X_1$	$\mathrm{Rot}_{xy}.\mathrm{Rot}_{xz}.\mathrm{Rot}_{yz}$	$\mathrm{Diag}(\lambda, \mu, \nu)$, $0 < \nu \leq \mu \leq \lambda$

Table 1.1

$$\mathrm{Rot}_{xy} = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ -\sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathrm{Rot}_{xz} = \begin{pmatrix} \cos(a) & 0 & \sin(a) \\ 0 & 1 & 0 \\ -\sin(a) & 0 & \cos(a) \end{pmatrix}, \mathrm{Rot}_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(a) & \sin(a) \\ 0 & -\sin(a) & \cos(a) \end{pmatrix}$$

$$\mathrm{Boost}_{xz} = \begin{pmatrix} \cosh(a) & 0 & \sinh(a) \\ 0 & 1 & 0 \\ \sinh(a) & 0 & \cosh(a) \end{pmatrix}, \mathrm{Boost}_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(a) & \sinh(a) \\ 0 & \sinh(a) & \cosh(a) \end{pmatrix}.$$

1.7.1 The abelian Lie group $\mathbb{R}^3$

We consider the 3-dimensional unimodular abelian Lie group $\mathbb{R}^3$ and $\mathbb{R}^3$ its Lie algebra. The Lie algebra $\mathbb{R}^3$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ such that

$$[X_1, X_2] = 0,\ [X_3, X_1] = 0 \quad \text{and} \quad [X_3, X_2] = 0.$$

Theorem 1.7.1 ([7]). *Let $\langle\ ,\ \rangle$ be a scalar product on $\mathbb{R}^3$. Then there exists an automorphism ϕ of $\mathbb{R}^3$ such that*

$$\mathrm{Mat}(\phi^*(\langle\ ,\ \rangle), \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

.

We have the set of homomorphisms of $\mathbb{R}^3$ is

$$\mathrm{H}(\mathbb{R}^3) = M_3(\mathbb{R}^3).$$

1.7.2 The Heisenberg group Nil

We consider the 3-dimensional unimodular Lie group H and we denote by $\mathfrak{n}$ its Lie algebra. The Lie algebra $\mathfrak{n}$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ such that

$$[X_1, X_2] = X_3,\ [X_3, X_1] = 0 \quad \text{and} \quad [X_3, X_2] = 0.$$

The following result will be useful later in chapter 3.

Theorem 1.7.2 ([7]). *Let $\langle\ ,\ \rangle$ be a scalar product on $\mathfrak{n}$. Then there exists an automorphism ϕ of $\mathfrak{n}$ such that*

$$\mathrm{Mat}(\phi^*(\langle\ ,\ \rangle), \mathbb{B}_0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $0 < \lambda$.

We determine now all the homomorphisms. Let $\xi : \mathfrak{n} \longrightarrow \mathfrak{n}$ be an homomorphism. Then ξ preserves $[\mathfrak{n}, \mathfrak{n}]$ and hence

$$\xi(X_1) = \alpha_1 X_1 + \beta_1 X_2 + \gamma_1 X_3,\ \xi(X_2) = \alpha_2 X_1 + \beta_2 X_2 + \gamma_2 X_3 \quad \text{and} \quad \xi(X_3) = \gamma X_3.$$

Then ξ is an homomorphism if and only if

$$\gamma = \alpha_1\beta_2 - \alpha_2\beta_1$$

Then, the set of homomorphisms of $\mathfrak{n}$ is

$$\mathrm{H}(\mathfrak{n}) = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \gamma_1 & \gamma_2 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}, \right\}.$$

1.7.3 The solvable Lie group $\widetilde{E_0}(2)$

We consider the 3-dimensional unimodular Lie group $E_0(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2, \mathbb{R})$ and we denote by $\mathfrak{g} = \mathbb{R}^2 \rtimes \mathrm{so}(2, \mathbb{R})$ its Lie algebra. The Lie algebra $\mathfrak{g}$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ such that

$$[X_1, X_2] = 0,\ [X_3, X_1] = X_2 \quad \text{and} \quad [X_3, X_2] = -X_1.$$

The following result will be useful later.

Theorem 1.7.3 ([7]). *Let $\langle\ ,\ \rangle$ be a scalar product on $\mathfrak{g}$. Then there exists an automorphism ϕ of $\mathfrak{g}$ such that*

$$\mathrm{Mat}(\phi^*(\langle\ ,\ \rangle), \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

where $0 < \mu \leq 1$ and $\nu > 0$.

We determine now all the homomorphisms. Let $\xi : \mathfrak{g} \longrightarrow \mathfrak{g}$ be an homomorphism. Then ξ preserves $[\mathfrak{g}, \mathfrak{g}]$ and hence

$$\xi(X_1) = \alpha_1 X_1 + \beta_1 X_2,\ \xi(X_2) = \alpha_2 X_1 + \beta_2 X_2 \quad \text{and} \quad \xi(X_3) = \alpha_3 X_1 + \beta_3 X_3 + \gamma X_3.$$

Then ξ is an homomorphism if and only if

$$\begin{cases} \alpha_1 - \gamma\beta_2 = 0, \\ \gamma\alpha_1 - \beta_2 = 0, \\ \alpha_2 + \beta_1\gamma = 0, \\ \gamma\alpha_2 + \beta_1 = 0. \end{cases}$$

We can deduce easily that the set of homomorphisms of $\mathfrak{g}$ is

$$\mathrm{H}(\mathfrak{g}) = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}, a, b, \alpha, \beta, \gamma \in \mathbb{R}, \gamma \neq 1 \right\}.$$

1.7.4 The solvable Lie group *Sol*

We consider the 3-dimensional unimodular Lie group Sol and we denote by $\mathfrak{sol} = \mathbb{R}^2 \rtimes \mathbb{R}$ its Lie algebra. The Lie algebra $\mathfrak{sol}$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ such that

$$[X_1, X_2] = 0, \ [X_3, X_1] = X_1 \quad \text{and} \quad [X_3, X_2] = -X_2.$$

The following result will be useful later.

Theorem 1.7.4 ([7]). *Let $\langle\ ,\ \rangle$ be a scalar product on $\mathfrak{sol}$. Then there exist two automorphisms ϕ_1 and ϕ_2 of $\mathfrak{sol}$ such that*

$$\mathrm{Mat}(\phi_1^*(\langle\ ,\ \rangle), \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

and

$$\mathrm{Mat}(\phi_2^*(\langle\ ,\ \rangle), \mathbb{B}_0) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

where $\mu > 1$ and $\nu > 0$.

We determine now all the homomorphisms. Let $\xi : \mathfrak{sol} \longrightarrow \mathfrak{sol}$ be an homomorphism. Then ξ preserves $[\mathfrak{sol}, \mathfrak{sol}]$ and hence

$$\xi(X_1) = \alpha_1 X_1 + \beta_1 X_2, \ \xi(X_2) = \alpha_2 X_1 + \beta_2 X_2 \quad \text{and} \quad \xi(X_3) = \alpha_3 X_1 + \beta_3 X_3 + \gamma X_3.$$

Then ξ is an homomorphism if and only if

$$\begin{cases} \alpha_1(\gamma - 1) = 0, \\ \beta_1(\gamma + 1) = 0, \\ \alpha_2(\gamma + 1) = 0, \\ \beta_2(\gamma - 1) = 0. \end{cases}$$

We can deduce easily that the set of homomorphisms of $\mathfrak{sol}$ is

$$\mathrm{H}(\mathfrak{sol}) = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & 0 & a \\ 0 & \beta & b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \beta & a \\ \alpha & 0 & b \\ 0 & 0 & -1 \end{pmatrix}, a, b, \alpha, \beta, \gamma \in \mathbb{R}, \gamma^2 \neq 1 \right\}.$$

1.7.5 The simple Lie group $\widetilde{PSL}(2, \mathbb{R})$

We consider the 3-dimensional unimodular Lie group $SL(2, \mathbb{R})$ and we denote by $sl(2, \mathbb{R})$ its Lie algebra. The Lie algebra $sl(2, \mathbb{R})$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ such that

$$[X_1, X_2] = 2X_3, \ [X_3, X_1] = 2X_2 \quad \text{and} \quad [X_3, X_2] = 2X_1.$$

The following result will be useful later.

Theorem 1.7.5 ([7]). *Let $\langle\ ,\ \rangle$ be a scalar product on $sl(2, \mathbb{R})$. Then there exists an automorphism ϕ of $sl(2, \mathbb{R})$ such that*

$$\mathrm{Mat}(\phi^*(\langle\ ,\ \rangle), \mathbb{B}_0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

where $0 < \nu \leq \mu$ and $\lambda > 0$.

1.7.6 The simple Lie group $SU(2)$

We consider the 3-dimensional unimodular Lie group $SU(2)$ and we denote by $su(2)$ its Lie algebra. The Lie algebra $su(2)$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ such that

$$[X_1, X_2] = X_3, \ [X_3, X_1] = X_2 \quad \text{and} \quad [X_3, X_2] = -X_1.$$

The following result will be useful later.

Theorem 1.7.6 ([7]). *Let $\langle\ ,\ \rangle$ be a scalar product on $su(2)$. Then there exists an automorphism ϕ of $su(2)$ such that*

$$\operatorname{Mat}(\phi^*(\langle\ ,\ \rangle), \mathbb{B}_0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

where $0 < \nu \leq \mu \leq \lambda$.

Chapter 2

Harmonic and biharmonic homomorphisms between Riemannian Lie groups

Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds with $m = \dim M$ and $n = \dim N$. We denote by ∇^M and ∇^N the Levi-Civita connexions associated respectively to g and h and by $T^\phi N$ the vector bundle over M pull-back of TN by ϕ. It is an Euclidean vector bundle and the tangent map of ϕ is a bundle homomorphism $d\phi : TM \longrightarrow T^\phi N$. Moreover, $T^\phi N$ carries a connexion ∇^ϕ pull-back of ∇^N by ϕ and there is a connexion on the vector bundle $\mathrm{End}(TM, T^\phi N)$ given by

$$(\nabla_X A)(Y) = \nabla^\phi_X A(Y) - A\left(\nabla^M_X Y\right), \quad X, Y \in \Gamma(TM), A \in \Gamma\left(\mathrm{End}(TM, T^\phi N)\right).$$

The map ϕ is called harmonic if it is a critical point of the energy $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \nu_g$. The corresponding Euler-Lagrange equation for the energy is given by the vanishing of the tension field

$$\tau(\phi) = \mathrm{tr}_g \nabla d\phi = \sum_{i=1}^{m} (\nabla_{E_i} d\phi)(E_i), \tag{2.1}$$

where $(E_i)_{i=1}^m$ is a local frame of orthonormal vector fields. Note that $\tau(\phi) \in \Gamma(T^\phi N)$. The map ϕ is called biharmonic if it is a critical point of the bienergy of ϕ defined by $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \nu_g$. The corresponding Euler-Lagrange equation for the bienergy is given by the vanishing of the bitension field

$$\tau_2(\phi) = -\mathrm{tr}_g (\nabla^\phi)^2_{.,.}\tau(\phi) - \mathrm{tr}_g R^N(\tau(\phi), d\phi(\,.\,))d\phi(\,.\,) = -\sum_{i=1}^{m}\left((\nabla^\phi)^2_{E_i,E_i}\tau(\phi) + R^N(\tau(\phi), d\phi(E_i))d\phi(E_i)\right),$$

(2.2)

where $(E_i)_{i=1}^m$ is a local frame of orthonormal vector fields, $(\nabla^\phi)^2_{X,Y} = \nabla^\phi_X \nabla^\phi_Y - \nabla^\phi_{\nabla^M_X Y}$ and R^N is the curvature of ∇^N given by

$$R^N(X,Y) = \nabla^N_X \nabla^N_Y - \nabla^N_Y \nabla^N_X - \nabla^N_{[X,Y]}.$$

2.1 General properties and first examples

Let (G,g) be a Riemannian Lie group, i.e., a Lie group endowed with a left invariant Riemannian metric. If $\mathfrak{g} = T_eG$ is its Lie algebra and $\langle\ ,\ \rangle_{\mathfrak{g}} = g(e)$ then there exists a unique bilinear map $A : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called the Levi-Civita product associated to $(\mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}})$ given by the formula:

$$2\langle A_u v, w\rangle_{\mathfrak{g}} = \langle [u,v]^{\mathfrak{g}}, w\rangle_{\mathfrak{g}} + \langle [w,u]^{\mathfrak{g}}, v\rangle_{\mathfrak{g}} + \langle [w,v]^{\mathfrak{g}}, u\rangle_{\mathfrak{g}}. \tag{2.3}$$

A is entirely determined by the following properties:

1. for any $u, v \in \mathfrak{g}$, $A_u v - A_v u = [u,v]^{\mathfrak{g}}$,

2. for any $u, v, w \in \mathfrak{g}$, $\langle A_u v, w\rangle_{\mathfrak{g}} + \langle v, A_u w\rangle_{\mathfrak{g}} = 0$.

If we denote by u^ℓ the left invariant vector field on G associated to $u \in \mathfrak{g}$ then the Levi-Civita connection associated to (G,g) satisfies $\nabla_{u^\ell} v^\ell = (A_u v)^\ell$. The couple $(\mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}})$ defines a vector say $U^{\mathfrak{g}} \in \mathfrak{g}$ by

$$\langle U^{\mathfrak{g}}, v\rangle_{\mathfrak{g}} = \mathrm{tr}(\mathrm{ad}_v), \quad \text{for any } v \in \mathfrak{g}. \tag{2.4}$$

One can deduce easily from (2.3) that, for any orthonormal basis $(e_i)_{i=1}^n$ of $\mathfrak{g}$,

$$U^{\mathfrak{g}} = \sum_{i=1}^n A_{e_i} e_i. \tag{2.5}$$

Note that $\mathfrak{g}$ is unimodular if and only if $U^{\mathfrak{g}} = 0$.

In the following, we give (G, g) and (H, h) two Riemannian Lie groups of respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\langle , \rangle_{\mathfrak{g}} = g_e$ and $\langle , \rangle_{\mathfrak{h}} = h_e$. Let ∇^G and ∇^H be the Levi-Civita connections on (G, g) and (H, h) and let A and B denote the Levi Civita products respectively on the Euclidean Lie algebras $(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$ and $(\mathfrak{h}, \langle , \rangle_{\mathfrak{h}})$. Let $\phi : (G, g) \longrightarrow (H, h)$ be a Lie group homomorphism between two Riemannian Lie groups. Let $\xi : \mathfrak{g} \longrightarrow \mathfrak{h}$ The differential of ϕ at e, it is a Lie algebra homomorphism. There is a left action of G on $\Gamma(T^{\phi}H)$ given by

$$(a.X)(b) = T_{\phi(ab)}L_{\phi(a^{-1})}X(ab), \quad a, b \in G, X \in \Gamma(T^{\phi}H).$$

A section X of $T^{\phi}H$ is called left invariant if, for any $a \in G$, $a.X = X$. For any left invariant section X of $T^{\phi}H$, we have for any $a \in G$, $X(a) = (X(e))^{\ell}(\phi(a))$. Thus the space of left invariant sections is isomorphic to the Lie algebra $\mathfrak{h}$. Since ϕ is a homomorphism of Lie groups and g and h are left invariant, one can see that $\tau(\phi)$ and $\tau_2(\phi)$ are left invariant and hence ϕ is harmonic (resp. biharmonic) iff $\tau(\phi)(e) = 0$ (resp. $\tau_2(\phi)(e) = 0$).

Remark 6. *The map $\mathfrak{h} \longrightarrow \mathbb{R}$ given by $u \longmapsto \mathrm{tr}(\xi^* \circ ad_u \circ \xi)$ defines a linear form on $\mathfrak{h}$, then there exists $U^{\xi} \in \mathfrak{h}$ such that for all $u \in \mathfrak{h}$:*

$$\langle U^{\xi}, u\rangle_{\mathfrak{h}} = \mathrm{tr}(\xi^* \circ ad_u \circ \xi)$$

Now, one can see that

$$\begin{cases} \tau(\xi) := \tau(\phi)(e) = U^{\xi} - \xi(U^{\mathfrak{g}}), \\ \tau_2(\xi) := \tau_2(\phi)(e) = -\displaystyle\sum_{i=1}^{n} \big(B_{\xi(e_i)}B_{\xi(e_i)}\tau(\xi) + K^H(\tau(\xi), \xi(e_i))\xi(e_i)\big) + B_{\xi(U^{\mathfrak{g}})}\tau(\xi), \end{cases} \tag{2.6}$$

where B is the Levi-Civita product associated to $(\mathfrak{h}, \langle\ ,\ \rangle_{\mathfrak{h}})$,we have

$$U^{\xi} = \sum_{i=1}^{n} B_{\xi(e_i)}\xi(e_i), \tag{2.7}$$

$(e_i)_{i=1}^n$ is an orthonormal basis of $\mathfrak{g}$ and K^H is the curvature of B given by $K^H(u, v) = [B_u, B_v] - B_{[u,v]}$. So we get the following proposition.

Proposition 2.1.1. *Let $\phi : G \longrightarrow H$ be an homomorphism between two Riemannian Lie groups. Then ϕ is harmonic (resp. biharmonic) iff $\tau(\xi) = 0$ (resp. $\tau_2(\xi) = 0$), where $\xi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is the differential of ϕ at e.*

Proposition 2.1.2. *Let $\phi : (G, g) \longrightarrow (H, h)$ be an homomorphism between two Riemannian Lie groups where H is abelian. Then ϕ is biharmonic and it is harmonic when $\mathfrak{g}$ is unimodular. In particular, any character $\mathcal{X} : G \longrightarrow \mathbb{R}$ is biharmonic and it is harmonic when $\mathfrak{g}$ is unimodular*

Proof. Since the Lie group H is abelian, then the Lie algebra $\mathfrak{h}$ is abelian then we have $[u, v]^{\mathfrak{h}} = 0$ for all $u, v \in \mathfrak{h}$. Thus:

$$2\langle B_u v, w\rangle^{\mathfrak{h}} = \langle [u, v], w\rangle^{\mathfrak{h}} + \langle [w, v], u\rangle^{\mathfrak{h}} + \langle [w, u], v\rangle^{\mathfrak{h}}$$

Then $B = 0$. The formula of the fields of bi-tension given in (2.6) gives that $\tau_2(\xi) = 0$ then ϕ is biharmonic. If we suppose that $\mathfrak{g}$ is unimodular then $tr(ad_u^{\mathfrak{g}}) = 0$ for all $u \in \mathfrak{g}$ then we have $U^{\mathfrak{g}} = 0$. Since $B = 0$ then $U^{\xi} = 0$. The formula of the fields of tension given in the (2.6) gives that $\tau_2(\xi) = 0$ then ϕ is harmonic. □

Lemma 2.1.1. *For all $u \in \mathfrak{h}$ we have the formula:*

$$\langle \tau_2(\xi), u\rangle_{\mathfrak{h}} = \mathrm{tr}(\xi^* \circ (ad_u + ad_u^*) \circ ad_{\tau(\xi)} \circ \xi) - \langle [u, \tau(\xi)]_{\mathfrak{h}}, \tau(\xi)\rangle_{\mathfrak{h}} - \langle [\tau(\xi), U^{\xi}]_{\mathfrak{h}}, u\rangle_{\mathfrak{h}}$$

Lemma 2.1.2. *For all $u \in K(h)$, $\langle U^{\xi}, u\rangle_{\mathfrak{h}} = 0$. In other words $U^{\xi} \in K(h)^{\perp}$*

Proposition 2.1.3. *Let $\phi : G \longrightarrow H$ be an homomorphism between two Riemannian Lie groups. Then:*
(i) If the metric on G is bi-invariant and ϕ is a submersion then ϕ is harmonic.
(ii) If the metric on H is bi-invariant then ϕ is biharmonic, it is harmonic when $\mathfrak{g}$ is unimodular.

Proof. For the point (i), since the metric g of G is bi-invariant then ad_u is skew-symmetric with respect to $\langle , \rangle_{\mathfrak{g}}$ for all $u \in \mathfrak{g}$ we have:

$$\langle A_u u, v\rangle_{\mathfrak{g}} = \langle [u, v], u\rangle_{\mathfrak{g}} = \langle ad_u v, u\rangle_{\mathfrak{g}} = -\langle v, ad_u u\rangle_{\mathfrak{g}} = 0.$$

Thus $A_u u = 0$ for all $u \in \mathfrak{g}$ and in particular $U^{\mathfrak{g}} = 0$. On the other hand let $v \in \mathfrak{h}$, since ϕ is a submersion then $\xi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is an homomorphism of Lie algebras surjective, then exists $u \in \mathfrak{g}$ such that $v = \xi(u)$ and $ad_v \circ \xi = ad_{\xi(u)} \circ \xi = \xi \circ ad_u$, which gives that:

$$\langle U^{\xi}, v\rangle_{\mathfrak{h}} = \mathrm{tr}(\xi^* \circ (ad_v) \circ \xi) = \mathrm{tr}(\xi^* \circ \xi \circ (ad_u))$$

since ad_u is skew-symmetric and $\xi^* \circ \xi$ is symmetric. Thus $U^{\xi} = 0$, hence $\tau(\xi) = 0$.
For the point (ii), by lemma 2.1.1 we have for all $u \in \mathfrak{h}$:

$$\langle \tau_2(\xi), u\rangle_{\mathfrak{h}} = \mathrm{tr}(\xi^* \circ (ad_u + ad_u^*) \circ ad_{\tau(\xi)} \circ \xi) - \langle [u, \tau(\xi)]_{\mathfrak{h}}, \tau(\xi)\rangle_{\mathfrak{h}} - \langle [\tau(\xi), U^{\xi}]_{\mathfrak{h}}, u\rangle_{\mathfrak{h}}$$

The metric on H is bi-invariant then $K(h) = \mathfrak{h}$ and hence, according to Lemma 2.1.2, $U^\xi = 0$. And we have

$$\langle [u, \tau(\xi)]_{\mathfrak{h}}, \tau(\xi)\rangle_{\mathfrak{h}} = -\langle ad_{\tau(\xi)} u, \tau(\xi)\rangle_{\mathfrak{h}} = \langle ad_{\tau(\xi)} \tau(\xi), u\rangle_{\mathfrak{h}} = 0$$

Thus $\tau_2(\xi) = 0$, hence ϕ is biharmonic. If $\mathfrak{g}$ is unimodular, then $U^{\mathfrak{g}} = 0$ and as $U^\xi = 0$ we conclude that $\tau(\xi) = 0$, thus ϕ is harmonic. $\square$

Example 22. *Let (G, g) be a Riemannian Lie group endowed with a bi-invariant metric g. Let N be a closed subgroup in G, then G/N has a unique Lie group structure such that the canonical projection $\pi : G \longrightarrow G/N$ is a submersion. By applying proposition (2.1.3), we obtain that for any left invariant metric h on G/N, the canonical projection $\pi : (G, g) \longrightarrow (G/N, h)$ is harmonic.*

Example 23. *Let G be a compact Lie group and $\rho : G \longrightarrow GL(V, \mathbb{R})$ a finite representation of G. Then there exists a definite positive product $\langle , \rangle$ on V which is G-invariant, thus $\rho : G \longrightarrow SO(V, \langle , \rangle)$. Now $SO(V, \langle , \rangle)$ has a bi-invariant Riemannian metric h and hence for any left invariant Riemannian metric g on G, $\rho : (G, g) \longrightarrow (SO(V, \langle , \rangle), h)$ is harmonic.*

2.2 Harmonic automorphisms of a Riemannian Lie group

In this section, we denote by $H(g)$ the set of $a \in G$ such that c_a is harmonic, i.e. that:

$$H(g) = \{a \in G, \tau(Ad_a) = 0\}.$$

Note that $H(g)$ is not in general a subgroup of G since the computations of two harmonic automorphisms is not necessarily harmonic.

Remark 7. *1. We have all isometry is harmonic then $I(g) \subset H(g)$.*

2. For all $a \in Z(G)$ we have $c_a = Id_G$ this gives in particular that $Z(G) \subset I(g) \subset H(g)$.

3. Let $\varphi, \psi : G \longrightarrow G$ be automorphisms of Riemannian Lie groups. We assume that φ is harmonic and ψ is an isometry. Then $\varphi \circ \psi$ and $\psi \circ \varphi$ are harmonics.

Proposition 2.2.1. *The set $H(g)$ is stable by the actions to the right and to the left of the group $I(g)$.*

As a consequence, we obtain that the quotient $H(g)/I(g)$ is defined.

Lemma 2.2.1. *If G is unimodular, then $I(g)$ is an open subset of $H(g)$. In particular, the quotient space $H(g)/I(g)$ is discrete.*

Corollary 2.2.1. *If G is compact, then $H(g)/I(g)$ is finite.*

Theorem 2.2.1 ([3]). *Let (G, g) be a connected Riemannian Lie group such that $H(g) = G$. Then the Riemannian metric g is bi-invariant.*

Theorem 2.2.2 ([3]). *If G is a connected Lie group which is abelian or 2-step nilpotent then $H(g) = I(g) = Z(G)$.*

2.2.1 The harmonic cone of left invariant Riemannian metric

Now, we consider the following problem: Let (G, g) Riemannian Lie group, we want to determine all the pairs (φ, h) such that $\varphi : (G, g) \longrightarrow (G, h)$ is harmonic where h is a left-invariant metric on G and φ is an automorphism of the Lie group G.

Proposition 2.2.2. *$\varphi : (G, g) \longrightarrow (G, h)$ is harmonic if and only if $Id_G : (G, g) \longrightarrow (G, \varphi^* h)$ is harmonic*

Proof. It is clear that $\psi : (G, \varphi^* h) \longrightarrow (G, h)$ is an isometry, and we have $\varphi = \psi \circ Id_G$. We then conclude from the point 3 in remark 7 the automorphism $\varphi : (G, g) \longrightarrow (G, h)$ is harmonic if and only if $Id_G : (G, g) \longrightarrow (G, \varphi^* h)$ is harmonic. $\square$

Let denote by $CH(g)$ the set of left invariant metrics $h \in \mathcal{M}^l(G)$ such that $Id_G : (G, g) \longrightarrow (G, h)$ is harmonic. The solution of the problem is equivalent to the determination of the group $Aut(G)$ and the set $CH(g)$ of the left invariant Riemannian metric h on G such that $Id_G : (G, g) \longrightarrow (G, h)$ is harmonic.

Proposition 2.2.3. *Let (G, g) be a Riemannian Lie group. Then $h \in CH(g)$ if and only if, for any $u \in \mathfrak{g}$,*

$$\operatorname{tr}(J \circ ad_u) = \operatorname{tr}(ad_{Ju}),$$

where $J : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a positive definite symmetric endomorphism given by $h_e(u, v) = g_e(Ju, v)$ for any $u, v \in \mathfrak{g}$.

Proof. Let $g_e = \langle , \rangle_1$, $h_e = \langle , \rangle_2$, A the Levi-Civita product of $(\mathfrak{g}, \langle , \rangle_1)$ and B the Levi-Civita product of $(\mathfrak{g}, \langle , \rangle_1)$. We have, for any $u \in \mathfrak{g}$, and for any orthonormal basis $(e_i)_{i=1}^n$ of $\langle , \rangle_1$:

$$\begin{aligned}\langle \tau(Id_{\mathfrak{g}}), u\rangle_2 &= \textstyle\sum_{i=1}^n \langle B_{e_i} e_i, u\rangle_2 - \sum_{i=1}^n \langle A_{e_i} e_i, u\rangle_2 \\ &= \textstyle\sum_{i=1}^n \langle [u, e_i] e_i, \rangle_2 - \sum_{i=1}^n \langle A_{e_i} e_i, Ju\rangle_1 \\ &= \textstyle\sum_{i=1}^n \langle J[u, e_i], e_i\rangle_1 - \sum_{i=1}^n \langle [Ju, e_i], e_i\rangle_1 \\ &= \mathrm{tr}(J \circ ad_u) - \mathrm{tr}(ad_{Ju}).\end{aligned}$$

□

Corollary 2.2.2. *$CH(g)$ is a convex cone which contains g.*

Definition 2.2.1. *We call $CH(g)$ the harmonic cone of g and $dimCH(g)$ the harmonic dimension of g, where $dimCH(g)$ is the dimension of the subspace spanned by $CH(g)$.*

Proposition 2.2.4. *Let (G, g) be a Riemannian Lie group. Then $CH(g) = \mathcal{M}^l(G)$ if and only if g is bi-invariant.*

Proof. Suppose that g is bi-invariant, Proposition (2.1.3) gives that for any left-invariant metric h, $Id_G : (G, g) \longrightarrow (G, h)$ is harmonic and we have that $CH(g) = \mathcal{M}^l(G)$. Conversely, we assume that $CH(g) = \mathcal{M}^l(G)$. Then for all $a \in G, c_a^*(g) \in CH(g)$ what is equivalent to saying according to proposition (2.2.2) that $Ad_a : (G, g) \longrightarrow (G, g)$ and consequently $a \in H(g)$. Thus $H(g) = G$ and Theorem (2.2.1) then gives that g is bi-invariant. □

Theorem 2.2.3. *Let (G, g) be a unimodular Riemannian Lie group. Then*

$$dimCH(g) = \frac{n(n-1)}{2} + dimK(g)$$

, where n is the dimension of the Lie group G.

Proof. The scalar product $\langle , \rangle_{\mathfrak{g}}$ of $\mathfrak{g}$ induces a scalar product $\langle , \rangle$ of $\mathfrak{g}l(\mathfrak{g})$ given by:

$$\langle A, B\rangle = \mathrm{tr}(A^* B)$$

Let define the linear map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}l(\mathfrak{g})$ given by $u \longmapsto ad_u + ad_u^*$. It is clear that $ker\varphi = K(g)$ and we have φ induces an injective linear map $\psi : \mathfrak{g}/K(g) \longrightarrow \mathfrak{g}l(\mathfrak{g})$ such that $\psi \circ \pi = \varphi$ with $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/K(g)$ the canonical projection. We give $J \in Sym^+(\mathfrak{g})$ then:

$$\mathrm{tr}(J^* \circ (ad_u + ad_u^*)) = \mathrm{tr}(J^* \circ ad_u) + \mathrm{tr}(J^* \circ ad_u^*) = \mathrm{tr}(J \circ ad_u) + \mathrm{tr}(ad_u \circ J) = 2\mathrm{tr}(J \circ ad_u).$$

Since G is unimodular then $\mathrm{tr}(ad_{Ju}) = 0$ and from proposition (2.2.3), we get that:

$$CH(g) = \{J \in Sym^+(\mathfrak{g}), \mathrm{tr}(J \circ ad_u) = 0 \quad for \quad all \quad u \in \mathfrak{g}\}.$$

This formula gives that $J \in CH(g)$ if and only if $\langle J, ad_u + ad_u^* \rangle = 0$, thus:

$$CH(g) = Sym^+(\mathfrak{g}) \cap (\psi(\mathfrak{g}/K(g)))^\perp$$

where $(\psi(\mathfrak{g}/K(g)))^\perp$ denotes the orthogonal of $(\psi(\mathfrak{g}/K(g)))$ for the scalar product $\langle , \rangle$. On the other hand, since $Sym^+(\mathfrak{g})$ is an open of $Sym(\mathfrak{g})$, then $CH(g)$ is an open of vector space $Sym(\mathfrak{g}) \cap (\psi(\mathfrak{g}/K(g)))^\perp$ then $CH(g)$ spanned this space.

$$\dim CH(g) = \dim Sym(\mathfrak{g}) \cap (\psi(\mathfrak{g}/K(g)))^\perp$$

since $\psi(\mathfrak{g}/K(g)) \subset Sym(\mathfrak{g})$, then $\mathfrak{g}l(\mathfrak{g}) = Sym(\mathfrak{g}) + (\psi(\mathfrak{g}/K(g)))^\perp$. Thus

$$\dim Sym(\mathfrak{g}) \cap (\psi(\mathfrak{g}/K(g)))^\perp = \dim Sym(\mathfrak{g}) + \dim(\psi(\mathfrak{g}/K(g)))^\perp - \dim \mathfrak{g}l(\mathfrak{g}) = \dim Sym(\mathfrak{g}) + \dim(\psi(\mathfrak{g}/K(g)))$$

Since ψ is injective, we get that $\dim(\psi(\mathfrak{g}/K(g))) = \dim(\mathfrak{g}/K(g)) = \dim \mathfrak{g} - \dim K(g)$. hence

$$\dim CH(g) = \frac{n(n-1)}{2} + \dim K(g)$$

□

2.3 Riemannian immersions between Riemannian Lie groups

Consider a homomorphism $\xi : (\mathfrak{g}, \langle , \rangle_{\mathfrak{g}}) \longrightarrow (\mathfrak{h}, \langle , \rangle_{\mathfrak{h}})$ and we assume that ξ is an isometry. Then ξ is injective and we put $\mathfrak{h}_0 := \xi(\mathfrak{g})$, it is clear that $\mathfrak{h}_0$ is a Lie subalgebra of $\mathfrak{h}$. The scalar product $\langle , \rangle_{\mathfrak{h}}$ induces by restriction a scalar product $\langle , \rangle_{\mathfrak{h}_0}$ on Lie algebra $\mathfrak{h}_0$. Then write:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_0^\perp$$

Then any element $w \in \mathfrak{h}$ decomposes in a unique way in the form $w = w^0 + w^\perp$ where $w^0 \in \mathfrak{h}_0$ and $w^\perp \in \mathfrak{h}_0^\perp$. In particular we obtain that for all $u, v \in \mathfrak{h}_0$

$$B_u v = (B_u v)^0 + (B_u v)^\perp$$

Proposition 2.3.1. *The Levi-Civita product on $(\mathfrak{h}_0, \langle , \rangle_{\mathfrak{h}_0})$ is given by the bilinear map $B^0 : \mathfrak{h}_0 \times \mathfrak{h}_0 \longrightarrow \mathfrak{h}_0, (u, v) \longmapsto (B_u v)^0$ And, we have for all $u, v \in \mathfrak{g}$:*

$$B^0_{\xi(u)} \xi(v) = \xi(A_u v)$$

The map $B^{\perp}:\mathfrak{h}_0\times\mathfrak{h}_0\longrightarrow\mathfrak{h}_0$ given by $(u,v)\longmapsto(B_uv)^{\perp}$ is symmetric bilinear and the vector $H^{\xi(\mathfrak{g})}=\sum_{i=1}^{m}B^{\perp}_{e_i}e_i$ does not depend on the orthonormal basis $(e_i)_{1\leq i\leq m}$ chosen on $\mathfrak{h}_0$.

Definition 2.3.1. *The vector $H^{\xi(\mathfrak{g})}$ is called the mean curvature vector of ξ. It is an element of $\xi(\mathfrak{g})^{\perp}$*

Proposition 2.3.2. *The mean curvature vector $H^{\xi(\mathfrak{g})}$ and the tension field $\tau(\xi)$ are equal.*

Proof. Let $(e_i)_{1\leq i\leq m}$ be a orthonormal basis of $(\mathfrak{g},\langle,\rangle_{\mathfrak{g}}$, so it is clear that the homomorphism $\xi:(\mathfrak{g},\langle,\rangle_{\mathfrak{g}})\longrightarrow(\mathfrak{h}_0,\langle,\rangle_{\mathfrak{h}_0})$ is an isomorphism of Euclidean spaces, and we have $(\xi(e_i)),1\leq i\leq m$ is an orthonormal basis of $(\mathfrak{h}_0,\langle,\rangle_{\mathfrak{h}_0})$. By the proposition 2.3.1 we have:

$$H^{\xi(\mathfrak{g})}=\sum_{i=1}^{m}B^{\perp}_{\xi(e_i)}\xi(e_i)=\sum_{i=1}^{m}B_{\xi(e_i)}\xi(e_i)-B^{0}_{\xi(e_i)}\xi(e_i)=\sum_{i=1}^{m}B_{\xi(e_i)}\xi(e_i)-\xi(A_{e_i}e_i)=\tau(\xi)$$

□

Proposition 2.3.3. *The homomorphism $\varphi:(G,g)\longrightarrow(H,h)$ is a Riemannian immersion if and only if $\xi:(\mathfrak{g},\langle,\rangle_{\mathfrak{g}})\longrightarrow(\mathfrak{h},\langle,\rangle_{\mathfrak{h}})$ is an isometry.*

Corollary 2.3.1. *Let $\varphi:G\longrightarrow H$ be a homomorphism between two Riemannian Lie groups which is also a Riemannian immersion. Then φ is harmonic if and only if $H^{\xi(\mathfrak{g})}=0$.*

Proposition 2.3.4. *Let $\varphi:G\longrightarrow H$ be a homomorphism between two Riemannian Lie groups. Suppose that φ is a Riemannian immersion, both $\mathfrak{g}$ and $\mathfrak{h}$ are unimodular and $\dim H=\dim G+1$. Then φ is harmonic.*

Proof. Since $\dim\mathfrak{h}=\dim\mathfrak{g}+1$ and $\xi:\mathfrak{g}\longrightarrow\mathfrak{h}$ is injective, then $\dim\xi(\mathfrak{g})^{\perp}=1$ hence $\xi(\mathfrak{g})^{\perp}=vect(f)$, and we have $H^{\xi(\mathfrak{g})}\in\xi(\mathfrak{g})^{\perp}$ then we get

$$H^{\xi(\mathfrak{g})}=\alpha f,\alpha\in\mathbb{R}$$

Choose an orthonormal basis $(e_1,...,e_n)$ of $\mathfrak{g}$ and complete by f to get an orthonormal basis $(\xi(e_1),...,\xi(e_n),f)$ of $\mathfrak{h}$. On the other hand we have $\mathfrak{g}$ and $\mathfrak{h}$ are unimodular then $U^{\mathfrak{g}}=0$ and $U^{\mathfrak{h}}=0$.

$$U^{\mathfrak{h}}=B_ff+\sum_{i=1}^{n}B_{\xi(e_i)}\xi(e_i)=B_ff+U^{\xi}$$

then we get $U^{\xi}=-B_ff$, then $\tau(\xi)=-B_ff$. Thus

$$\langle\tau(\xi),\tau(\xi)\rangle_{\mathfrak{h}}=\langle -B_ff,\alpha f\rangle_{\mathfrak{h}}=-\alpha\langle B_ff,f\rangle_{\mathfrak{h}}=0$$

Then φ is harmonic. □

Proposition 2.3.5. *Let $\varphi : G \longrightarrow H$ be a homomorphism between two Riemannian Lie groups. Suppose that φ is a Riemannian immersion, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, any derivation of $\mathfrak{g}$ is inner and $\xi(\mathfrak{g})$ is an ideal of $\mathfrak{h}$. Then φ is harmonic.*

Proof. Since $\xi(\mathfrak{g})$ is an ideal of $\mathfrak{h}$ we obtain that the restriction $\tilde{ad}_u$ of ad_u on $\xi(\mathfrak{g})$ defines a derivation of $\xi(\mathfrak{g})$ for all $u \in \mathfrak{h}$. On the other hand, $\xi : \mathfrak{g} \longrightarrow \xi(\mathfrak{g})$ is an isomorphism of Lie algebras and since any derivation of $\mathfrak{g}$ is inner then any derivation of $\xi(\mathfrak{g})$ is inner. Then denote by B^0 the Levi-Civita product on $(\xi(\mathfrak{g}), \langle , \rangle_{\mathfrak{h}})$ and fix an orthonormal basis $\{e_1, ..., e_n\}$ on $\mathfrak{g}$, we get that $\{\xi(e_1), ..., \xi(e_n)\}$ is an orthonormal basis of $\xi(\mathfrak{g})$. Thus for all $u \in \xi(\mathfrak{g})^{\perp}$, we have:

$$\langle H^{\xi(\mathfrak{g})}, u\rangle_{\mathfrak{h}} = \sum_{i=1}^{n} \langle B_{\xi(e_i)}\xi(e_i) - B^0_{\xi(e_i)}\xi(e_i), u\rangle_{\mathfrak{h}} = \sum_{i=1}^{n} \langle B_{\xi(e_i)}\xi(e_i), u\rangle_{\mathfrak{h}} = \sum_{i=1}^{n} \langle ad_u\xi(e_i), \xi(e_i)\rangle_{\mathfrak{h}} = \mathrm{tr}(\tilde{ad}_u).$$

Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ then $[\xi(\mathfrak{g}), \xi(\mathfrak{g})] = \xi(\mathfrak{g})$, we have $\tilde{ad}_u$ is an inner derivation of $\xi(\mathfrak{g})$, then $\tilde{ad}_u = ad_{[u_1,u_2]}$ with $u_1, u_2 \in \xi(\mathfrak{g})$ and therefore $\mathrm{tr}\tilde{ad}_u = 0$ for all $u \in \xi(\mathfrak{g})^{\perp}$. We then conclude that $H^{\xi(\mathfrak{g})} = 0$, Corollary (2.3.1) gives that this is equivalent to saying that φ is harmonic. □

2.4 Harmonic and Biharmonic submersions between Riemannian Lie groups

Let $\varphi : (G, g) \longrightarrow (H, h)$ be a homomorphism of Riemannian Lie groups. Recall that φ is a submersion if $T_a\varphi$ is surjective for all $a \in G$.

Proposition 2.4.1. *The homomorphism $\varphi : (G, g) \longrightarrow (H, h)$ is a submersion if and only if the homomorphism $\xi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is surjective.*

Proof. It is clear by the definition of submersion that if φ is a submersion, then ξ is surjective. Conversely, we suppose that $\xi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is surjective, it is clear that for all $a \in G$ we have $\varphi \circ L_a = L_{\varphi(a)} \circ \varphi$, passaging to the differential that $T_a\varphi \circ T_eL_a = T_eL_{\varphi(a)} \circ \xi$. Thus $T_a\varphi$ is surjective for all $a \in G$, hence φ is a submersion. □

In the following we suppose that $\varphi(G, g) \longrightarrow (H, h)$ is a submersion. Let $G_0 = \ker\varphi$ and $\mathfrak{g}_0 = \ker\xi$. Since G_0 is a normal subgroup in G then G/G_0 is a Lie group and φ induces a homomorphism of Lie groups $\bar{\varphi} : G/G_0 \longrightarrow H$. Moreover, $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{g}_0$ has a Lie algebra structure such that the canonical projection $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}_0$ is a homomorphism of Lie algebras. Finally, we put $\bar{\xi} := T_e\bar{\varphi}$.

Proposition 2.4.2. *Lie algebras $Lie(G/G_0)$ and $\mathfrak{g}/\mathfrak{g}_0$ are isomorphic*

Proof. The canonical projection $p : G \longrightarrow G/G_0$ is a homomorphism of Lie groups, we obtain a homomorphism of Lie algebras $\phi : \mathfrak{g} \longrightarrow Lie(G/G_0)$ by setting $\phi := T_e p$. On the other hand we have $\ker\phi = \mathfrak{g}_0$, since if $v \in \mathfrak{g}_0$ then $\exp(tv) \in G_0$ for all $t \in \mathbb{R}$ then:

$$\exp(t\phi(v)) = p \circ \exp(tv) = e$$

By taking the derivative at $t = 0$ of the previous formula we get that $\phi(v) = 0$ then $v \in \ker\phi$. Conversely, if $v \in \ker\phi$ then $\phi(v) = 0$ hence $\exp(t\phi(v)) = p \circ \exp(tv) = e$ for all $t \in \mathbb{R}$. That is to say that $exp(tv) \in G_0$ for all $t \in \mathbb{R}$ then $v \in \mathfrak{g}_0$. In summary we have $\ker\phi = \mathfrak{g}_0$ then ϕ induces an injective homomorphism of Lie algebras $\bar{\phi} : \mathfrak{g}/\mathfrak{g}_0 \longrightarrow Lie(G/G_0)$. We have $\dim \mathfrak{g}/\mathfrak{g}_0 = \dim Lie(G/G_0)$. Thus $\bar{\phi}$ is an isomorphism of Lie algebras. □

Denoted by $\bar{\xi} : \mathfrak{g}/\mathfrak{g}_0 \longrightarrow \mathfrak{h}$ then we have:

Proposition 2.4.3. *1. The homomorphism of Lie algebras ξ factors in the form $\xi = \bar{\xi} \circ \pi$.*

2. The linear map $\bar{\xi} : \mathfrak{g}/\mathfrak{g}_0 \longrightarrow \mathfrak{h}$ is an isomorphism of Lie algebras.

We denote by $r : \mathfrak{g}/\mathfrak{g}_0 \longrightarrow \mathfrak{g}_0^\perp$ the inverse of , the restriction of π to $\mathfrak{g}_0^\perp$. Thus $r^*\langle,\rangle_{\mathfrak{g}}$ is an Euclidean product on $\mathfrak{g}/\mathfrak{g}_0$ which defines a left invariant Riemannian metric g_0 on G/G_0.

Proposition 2.4.4. *With the notations above, we have*

$$\tau(\xi) = \tau(\bar{\xi}) - \xi(H^{\ker\xi}),$$

where $\bar{\xi} : (\mathfrak{g}/\mathfrak{g}_0, r^*\langle,\rangle_{\mathfrak{g}}) \longrightarrow (\mathfrak{h}, \langle,\rangle_{\mathfrak{h}})$ $H^{\ker\xi}$ is the mean curvature vector of the inclusion of $\ker\xi$ *in $(\mathfrak{g}, \langle,\rangle_{\mathfrak{g}})$.*

Proof. We have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0^\perp$. Choose an orthonormal basis $(f_i)_{i=1}^p$ of $\mathfrak{g}_0$ and an orthonormal basis $(e_i)_{i=1}^q$ of $\mathfrak{g}_0^\perp$. If A and B denote the Levi-Civita products of $\mathfrak{g}$ and $\mathfrak{h}$ respectively, we have:

$$\tau(\xi) = \sum_{i=1}^{q} B_{\xi(e_i)}\xi(e_i) - \sum_{i=1}^{q} \xi(A_{e_i}e_i) - \sum_{i=1}^{p} \xi(A_{f_i}f_i).$$

Let A^0 be the Levi-Civita product of $\mathfrak{g}_0$, we get

$$\xi(H^{\ker\xi}) = \xi\left(\sum_{i=1}^{p}(A_{f_i}f_i)^\perp\right) = \sum_{i=1}^{p} \xi(A_{f_i}f_i - A^0_{f_i}f_i)$$

On the other hand, the canonical projection $\pi_{\mathfrak{g}_0^\perp} : (\mathfrak{g}_0^\perp, \langle , \rangle_{\mathfrak{g}}) \longrightarrow (\mathfrak{g}/\mathfrak{g}_0, r^*\langle , \rangle_{\mathfrak{g}})$ is an isometry then $(\pi(e_i))_{i=1}^q$ is an orthonormal basis of $(\mathfrak{g}/\mathfrak{g}_0, r^*\langle , \rangle_{\mathfrak{g}})$. Since $\xi = \bar{\xi} \circ \pi$, we get that:

$$U^\xi = \sum_{i=1}^q B_{\xi(e_i)}\xi(e_i) = \sum_{i=1}^q B_{\bar{\xi}(\pi(e_i))}\bar{\xi}(\pi(e_i)) = U^{\bar{\xi}}$$

Thus

$$\tau(\xi) = U^{\bar{\xi}} - \xi\left(\sum_{i=1}^q A_{e_i}e_i\right) - \xi(H^{\mathfrak{g}_0}$$

To finish, it suffices to show that $\bar{\xi}(U^{\mathfrak{g}/\mathfrak{g}_0}) = \xi\left(\sum_{i=1}^q A_{e_i}e_i\right)$, but this comes from the fact that $\pi(A_u v) = \bar{A}_{\pi(u)}\pi(v)$ for all $u, v \in \mathfrak{g}_0^\perp$ because $\pi_{\mathfrak{g}_0^\perp} : (\mathfrak{g}_0^\perp, \langle , \rangle_{\mathfrak{g}}) \longrightarrow (\mathfrak{g}/\mathfrak{g}_0, r^*\langle , \rangle_{\mathfrak{g}})$ is an isometry with $\bar{A}$ is the Levi Civita product on $(\mathfrak{g}/\mathfrak{g}_0, r^*\langle , \rangle_{\mathfrak{g}})$ so:

$$\bar{\xi}(U^{\mathfrak{g}/\mathfrak{g}_0}) = \sum_{i=1}^q \bar{\xi}(\bar{A}_{\pi(e_i)}\pi(e_i)) = \sum_{i=1}^q \bar{\xi} \circ \pi(A_{e_i}e_i) = \sum_{i=1}^q \xi(A_{e_i}e_i)$$

□

Corollary 2.4.1. *Let $\tau_2(\bar{\xi})$ be the bitension field of $\bar{\xi} : (\mathfrak{g}/\mathfrak{g}_0, r^*\langle , \rangle_{\mathfrak{g}}) \longrightarrow (\mathfrak{h}, \langle , \rangle_{\mathfrak{h}})$. Then we have:*

$$\tau_2(\xi) - \tau_2(\bar{\xi}) = \sum_{i=1}^n (B_{\xi(e_i)}B_{\xi(e_i)}\xi(H^{\ker\xi}) + K_{\mathfrak{h}}(\xi(H^{\ker\xi}), \xi(e_i))\xi(e_i)) - B_{\xi(U^{\mathfrak{g}})}\xi(H^{\ker\xi}).$$

where $(e_i)_{i=1}^n)$ is an orthonormal basis of $(\mathfrak{g}, \langle , \rangle)$.

Proposition 2.4.5. *Let $\varphi : (G, g) \longrightarrow (H, h)$ be a submersion between two Riemannian Lie groups. Then:*
(i) If $\ker\xi$ is minimal then φ is harmonic (resp. biharmonic) if and only if $\bar{\varphi}$ is harmonic (resp. biharmonic).
(ii) If $\bar{\varphi}$ is harmonic then φ is harmonic if and only if $\ker\xi$ is minimal.

Let $\varphi : (G, g) \longrightarrow (H, h)$ be a submersion between two connected Riemannian Lie groups. Then we have $\bar{\varphi} : G/G_0 \longrightarrow H$ is an isomorphism. If we endow G/G_0 with the left invariant metric $\bar{\varphi}^*h$ we obtain then that $\bar{\varphi} : (G/G_0, \bar{\varphi}^*h) \longrightarrow (H, h)$ is an isometry. So φ is harmonic (resp. biharmonic) if and only if $p : (G, g) \longrightarrow (G/G_0, \varphi^*h)$ is harmonic (resp. biharmonic). So the study of harmonic or biharmonic submersion between two connected Riemannian Lie groups is equivalent to the study of the projections $p : (G, g) \longrightarrow (G/G_0, h)$ where (G, g) is a connected Lie group, G_0 is a normal subgroup and h is left invariant Riemannian metric on

G/G_0. To build harmonic or biharmonic such projections, let first understand how G can be constructed from G/G_0 and G_0. Let G_0 is a normal subgroup of G, and $p : (G, g) \longrightarrow (H, h)$ where $H = G/G_0$, denote by $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}_0$ the natural projection and $r : \mathfrak{g}/\mathfrak{g} \longrightarrow \mathfrak{g}_0^{\perp}$ the inverse of the restriction of π to $\mathfrak{g}_0^{\perp}$, we have $\pi = T_e p$. Denoted by $\langle , \rangle^{\pi} = r^* \langle , \rangle_{\mathfrak{g}}$.

Corollary 2.4.2. *In the previous notations, we have the formula:*

$$\tau(\pi) = \tau(Id_{\mathfrak{h}}) - \pi(H^{\mathfrak{g}_0})$$

where $\tau(Id_{\mathfrak{h}})$ is the tension field of the map $Id_{\mathfrak{h}} : (h, \langle , \rangle^{\pi}) \longrightarrow (\mathfrak{h}, \langle , \rangle_{\mathfrak{h}})$

Corollary 2.4.3. *In the previous notations, we have the formula:*

$$\tau_2(\pi) = \tau_2(Id_{\mathfrak{h}}) + \sum_{i=1}^{n} (B_{\pi(e_i)} B_{\pi(e_i)} \pi(H^{\mathfrak{g}_0}) + K_{\mathfrak{h}}(\pi(H^{\mathfrak{g}_0}), \pi(e_i))\pi(e_i)) - B_{\pi(U^{\mathfrak{g}})} \pi(H^{\mathfrak{g}_0}).$$

where $(e_i)_{i=1}^{n})$ is an orthonormal basis of $(\mathfrak{g}, \langle , \rangle)$ and $\tau_2(Id_{\mathfrak{h}})$ is the bitension field of the map $Id_{\mathfrak{h}} : (h, \langle , \rangle^{\pi}) \longrightarrow (\mathfrak{h}, \langle , \rangle_{\mathfrak{h}})$.

Denote for all $u \in \mathfrak{g}$, $\tilde{ad}_u$ the restriction of ad_u to $\mathfrak{g}_0$, since $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$ then $\tilde{ad}_u \in Der(\mathfrak{g}_0)$. Then define $\rho : \mathfrak{h} \longrightarrow Der(\mathfrak{g}_0)$ and $\omega : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{g}_0$ given by:

$$\rho(v) = \tilde{ad}_{r(v)} \quad \text{and} \quad \omega(v_1, v_2) = [r(v_1), r(v_2)]_{\mathfrak{g}} - r([v_1, v_2]_{\mathfrak{h}})$$

Proposition 2.4.6. *In the previous notations, we have:*

1. *For all $v_1, v_2 \in \mathfrak{h}$ we have the formula:*

$$\rho([v_1, v_2]_{\mathfrak{h}}) = [\rho(v_1), \rho(v_2)] - \tilde{ad}_{\omega(v_1, v_2)}.$$

2. *Let $d_\rho \omega(v_1, v_2, v_3) = \oint (\rho(v_1)(\omega(v_2, v_3)) - \omega([v_1, v_2]_{\mathfrak{h}}, v_3))$, alors $d_\rho \omega = 0$.*

Proposition 2.4.7 ([3]). *For all $v \in \mathfrak{h}$ we have*

$$\langle \pi(H^{\mathfrak{g}_0}), v \rangle^{\pi} = \mathrm{tr} \rho(v)$$

Proposition 2.4.8 ([3]). *Let G be a connected Riemannian Lie group and G_0 a semisimple normal subgroup of G. Then $G_0 \hookrightarrow G$ is minimal and $p : G \longrightarrow G/G_0$ is harmonic when G/G_0 is endowed with the quotient metric g_0. Moreover, for any left invariant Riemannian metric h on G/G_0, $p : (G, g) \longrightarrow (G/G_0, h)$ is harmonic (resp. biharmonic) if and only if $Id_{G/G_0} : (G/G_0, g_0) \longrightarrow (G/G_0, h)$ is harmonic (resp. biharmonic).*

2.5 When harmonicity and biharmonicity are equivalent

We give now, the study of situations where harmonicity and bi-harmonicity are equivalent in the context of Riemannian Lie groups.

Theorem 2.5.1. *Let $\varphi : G \longrightarrow H$ be a homomorphism between two Riemannian Lie groups such that the sectional curvature of (H, h) is non-positive and $\mathfrak{g}$ is unimodular. Then φ is harmonic if and only if it is biharmonic.*

Proof. Let $(e_i)_{i=1}^n$ be an orthonormal basis of $(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$, by the formula (2.6) we have

$$\langle \tau_2(\xi), \tau(\xi)\rangle_{\mathfrak{h}} = -\sum_{i=1}^{n}(\langle B_{\xi(e_i)}B_{\xi(e_i)}\tau(\xi), \tau(\xi)\rangle_{\mathfrak{h}} + \langle K^{\mathfrak{h}}(\tau(\xi), \xi(e_i))\xi(e_i), \tau(\xi)\rangle_{\mathfrak{h}} + \langle B_{\xi(U^{\mathfrak{g}})}\tau(\xi), \tau(\xi)\rangle_{\mathfrak{h}}$$

$$= \sum_{i=1}^{n}(\langle B_{\xi(e_i)}\tau(\xi), B_{\xi(e_i)}\tau(\xi)\rangle_{\mathfrak{h}} - \langle K^{\mathfrak{h}}(\xi(e_i), \tau(\xi))\xi(e_i), \tau(\xi)\rangle_{\mathfrak{h}}$$

If φ is biharmonic then $\tau_2(\xi) = 0$, then we have:

$$\sum_{i=1}^{n}(\langle B_{\xi(e_i)}\tau(\xi), B_{\xi(e_i)}\tau(\xi)\rangle_{\mathfrak{h}} = \sum_{i=1}^{n}\langle K^{\mathfrak{h}}(\xi(e_i), \tau(\xi))\xi(e_i), \tau(\xi)\rangle_{\mathfrak{h}}$$

Since the sectional curvature of (H, h) is negative then $\langle K^{\mathfrak{h}}(\xi(e_i), \tau(\xi))\xi(e_i), \tau(\xi)\rangle_{\mathfrak{h}} \leq 0$ for all $i = 1, n$. Thus $\sum_{i=1}^{n}(\langle B_{\xi(e_i)}\tau(\xi), B_{\xi(e_i)}\tau(\xi)\rangle_{\mathfrak{h}} = 0$, hence $B_{\xi(e_i)}\tau(\xi) = 0$ for all $1 \leq i \leq n$. We have $\mathfrak{g}$ is unimodular then $U^{\mathfrak{g}} = 0$, then $\tau(\xi) = U^{\xi}$.We have

$$\langle \tau(\xi), \tau(\xi)\rangle_{\mathfrak{h}} = \langle U^{\xi}, \tau(\xi)\rangle_{\mathfrak{h}} = \sum_{i=1}^{n}\langle B_{\xi(e_i)}\xi(e_i), \tau(\xi)\rangle = -\sum_{i=1}^{n}\langle \xi(e_i), B_{\xi(e_i)}\tau(\xi)\rangle = 0$$

Thus $\tau(\xi) = 0$, hence φ is harmonic. $\square$

Corollary 2.5.1. *Let $\varphi : G \longrightarrow H$ be a homomorphism between two Riemannian Lie groups such that the sectional curvature of (H, h) is non-positive and the sectional curvature of (G, g) is non-negative. Then φ is harmonic if and only if it is biharmonic.*

Proof. This is a consequence of Theorem 2.4.1 and the fact that a Lie group which admits a left invariant Riemannian metric with non-negative Ricci curvature must be unimodular. $\square$

Other situations where harmonic and bi-harmonic are equivalent are presented by the following theorem proved in a more general sitting by Oniciuc [8] in Propositions 2.2, 2.4, 2.5, 4.3.

Theorem 2.5.2 ([3]). *Let $\varphi : G \longrightarrow H$ be a homomorphism between two Riemannian Lie groups. In each for the following cases, φ is biharmonic if and only if it is harmonic:*

1. *The sectional curvature of (H, h) is non-positive and φ is a Riemannian immersion.*
2. *The sectional curvature of (H, h) is non-positive, φ is a Riemannian immersion and $dimH = dimG + 1$.*
3. *The sectional curvature of (H, h) is negative and $rank\xi > 1$.*
4. *The sectional curvature of (H, h) is negative and φ is a Riemannian submersion.*

Theorem 2.5.3 ([3]). *Let $\varphi : G \longrightarrow H$ be a homomorphism between two Riemannian Lie groups. In the following cases the harmonicity of φ and its biharmonicity are equivalent:*

1. *H is 2-step nilpotent and $\mathfrak{g}$ is unimodular.*
2. *φ is a Riemannian submersion and $\mathfrak{g}$ is unimodular.*
3. *φ is a Riemannian submersion, $ker\xi$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{g}$ is unimodular.*
4. *φ is a Riemannian submersion, $ker\xi$ is unimodular, $\dim H = 2$ and H is non abelian.*

We give now a criteria which will be useful in order to show that an homomorphism is harmonic if and only if it is biharmonic.

Let $\xi : (\mathfrak{g}, \langle\ ,\ \rangle_1) \longrightarrow (\mathfrak{h}, \langle\ ,\ \rangle_2)$ be an homomorphism. We suppose that $\mathfrak{g}$ is unimodular. The following formulas was established in Lemma 2.1.1 and remark 6:

$$\begin{aligned} \langle \tau(\xi), u\rangle_2 &= \mathrm{tr}(\xi^* \circ \mathrm{ad}_u \circ \xi), \\ \langle \tau_2(\xi), u\rangle_2 &= \mathrm{tr}(\xi^* \circ (\mathrm{ad}_u + \mathrm{ad}_u^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [u, \tau(\xi)], \tau(\xi)\rangle_2, \end{aligned}$$

where $\xi^* : \mathfrak{h} \longrightarrow \mathfrak{g}$ and $\mathrm{ad}_u^* : \mathfrak{h} \longrightarrow \mathfrak{h}$ are given by

$$\langle \xi^* u, v\rangle_1 = \langle u, \xi v\rangle_2 \quad \text{and} \quad \langle \mathrm{ad}_u^* x, y\rangle_2 = \langle \mathrm{ad}_u y, x\rangle_2,\ x, y, u \in \mathfrak{h}, v \in \mathfrak{g}.$$

By combining these two formulas, we get

$$\langle \tau_2(\xi), u\rangle_2 = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_u + \mathrm{ad}_u^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \mathrm{tr}(\xi^* \circ \mathrm{ad}_{[u,\tau(\xi)]} \circ \xi).$$

So if ξ is biharmonic then $\tau(\xi)$ is solution of the linear system

$$\mathrm{tr}(\xi^* \circ (\mathrm{ad}_u + \mathrm{ad}_u^*) \circ \mathrm{ad}_X \circ \xi) - \mathrm{tr}(\xi^* \circ \mathrm{ad}_{[u,X]} \circ \xi) = 0,\ u \in \mathfrak{h}. \tag{2.8}$$

If $\mathbb{B} = (f_1, \ldots, f_m)$ is a basis of $\mathfrak{h}$ this system is equivalent to

$$M_\xi(\mathbb{B})X = 0,$$

where $M_\xi(\mathbb{B}) = (m_{ij})_{1 \leq i \leq j \leq m}$ and

$$m_{ij} = \operatorname{tr}(\xi^* \circ (\mathrm{ad}_{f_i} + \mathrm{ad}^*_{f_i}) \circ \mathrm{ad}_{f_j} \circ \xi) - \operatorname{tr}(\xi^* \circ \mathrm{ad}_{[f_i, f_j]} \circ \xi).$$

We call $M_\xi(\mathbb{B})$ the test matrix of ξ in the basis $(f_1, \ldots, f_n)$.

Proposition 2.5.1. *If* $\det(M_\xi(\mathbb{B})) \neq 0$ *then* ξ *is biharmonic if and only if it is harmonic.*

Chapter 3

Biharmonic and harmonic homomorphisms between Riemannian three dimensional unimodular Lie groups

Thus the study of biharmonic and harmonic homomorphisms between connected and simply-connected Lie groups reduces to the study of their differential so, through this chapter, we consider homomorphisms $\xi : (\mathfrak{g}, \langle\ ,\ \rangle_1) \longrightarrow (\mathfrak{g}, \langle\ ,\ \rangle_2)$ where $\mathfrak{g}$ is a Lie algebra and $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are two Euclidean products. We call ξ harmonic (resp. biharmonic) if $\tau(\xi) = 0$ (resp. $\tau_2(\xi) = 0$).

The classification of biharmonic and harmonic homomorphisms will be done up to a conjugation. Two homomorphisms between Euclidean Lie algebras

$$\xi_1 : (\mathfrak{g}, \langle\ ,\ \rangle_1^1) \longrightarrow (\mathfrak{g}, \langle\ ,\ \rangle_2^1) \quad \text{and} \quad \xi_2 : (\mathfrak{g}, \langle\ ,\ \rangle_1^2) \longrightarrow (\mathfrak{g}, \langle\ ,\ \rangle_2^2)$$

are conjugate if there exists two isometric automorphisms $\phi_1 : (\mathfrak{g}, \langle\ ,\ \rangle_1^1) \longrightarrow (\mathfrak{g}, \langle\ ,\ \rangle_1^2)$ and $\phi_2 : (\mathfrak{g}, \langle\ ,\ \rangle_2^1) \longrightarrow (\mathfrak{g}, \langle\ ,\ \rangle_2^2)$ such that $\xi_2 = \phi_2 \circ \xi_1 \circ \phi_1^{-1}$.

In the following sections, the computation of $\tau(\xi)$ and $\tau_2(\xi)$ are performed by the software Maple and all the direct computations as well.

3.1 Harmonic and biharmonic homomorphisms on the 3-dimensional Heisenberg Lie group

The following result gives a complete classification of harmonic and biharmonic homomorphisms of $\mathfrak{n}$.

Theorem 3.1.1. *An homomorphism of $\mathfrak{n}$ is biharmonic if and only if it is harmonic. Moreover, it is harmonic if and only if it is conjugate to $\xi : (\mathfrak{n}, \langle\ ,\ \rangle_1) \longrightarrow (\mathfrak{n}, \langle\ ,\ \rangle_2)$ where*

$$\xi = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ 0 & 0 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} a\beta_3 & -a\alpha_3 & 0 \\ b\beta_3 & -b\alpha_3 & 0 \\ \alpha_3 & \beta_3 & 0 \end{pmatrix}, (\alpha_3, \beta_3) \neq (0,0)$$

and $\mathrm{Mat}(\langle\ ,\ \rangle_i, \mathbb{B}_0) = \mathrm{Diag}(\lambda_i, \lambda_i, 1)$ *and* $\lambda_i > 0$*,* $i = 1, 2$*.*

Proof. The first part of the theorem is a consequence of Theorem 2.5.3. On the other hand, according to Table 1.1, and homomorphism $\xi : (\mathfrak{n}, \langle\ ,\ \rangle_1) \longrightarrow (\mathfrak{n}, \langle\ ,\ \rangle_2)$ has, up to a conjugation, the form

$$\xi = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix} \quad \text{and} \quad \langle\ ,\ \rangle_i = \mathrm{Diag}(\lambda_i, \lambda_i, 1),\ \lambda_i > 0, i = 1, 2.$$

Then

$$\tau(\xi) = \frac{\alpha_3\beta_1 + \beta_3\beta_2}{\lambda_2\lambda_1} X_1 - \frac{\alpha_3\alpha_1 + \beta_3\alpha_2}{\lambda_2\lambda_1} X_2$$

and the second part of the theorem follows. □

3.2 Harmonic and biharmonic homomorphisms on $\widetilde{E_0}(2)$

The situation on $e_0(2)$ is different and there exists biharmonic homomorphisms which are not harmonic. The following two theorems give a complete classification of harmonic and biharmonic homomorphisms on $e_0(2)$.

Theorem 3.2.1. *An homomorphism of $e_0(2)$ is harmonic if and only if it it is conjugate to*

$\xi : (e_0(2), \langle\ ,\ \rangle_1) \longrightarrow (e_0(2), \langle\ ,\ \rangle_2)$ *where* $\mathrm{Mat}(\langle\ ,\ \rangle_i, \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_i & 0 \\ 0 & 0 & \nu_i \end{pmatrix}, 0 < \mu_i \leq 1, \nu_i > 0$ *and*

either

1. $\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}$ *with* $\gamma^2 \neq 1$ *and* $(a \neq 0, b \neq 0, \gamma = 0, \mu_2 = 1)\,,\ ((a,b) \neq (0,0), ab = 0, \gamma = 0)$ *or* $\quad (a = b = 0)$

2. $\xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}$ *and* $(a = b = 0, \alpha = 0), (a = b = 0, \beta = 0), (a = b = 0, \mu_1 = 1)$ *or* $(a = b = 0, \mu_2 = 1)$,

3. $\xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}$ *and* $(a = b = 0, \alpha = 0), (a = b = 0, \beta = 0), (a = b = 0, \mu_1 = 1)$ *or* $(a = b = 0, \mu_2 = 1)$.

Proof. According to Table 1.1, and homomorphism $\xi : (e_0(2), \langle\ ,\ \rangle_1) \longrightarrow (e_0(2), \langle\ ,\ \rangle_2)$ has, up to a conjugation, the form $\mathrm{Mat}(\langle\ ,\ \rangle_i, \mathbb{B}_0) = \mathrm{Diag}(1, \mu_i, \nu_i), i = 1, 2, 0 < \mu_i \leq 1, \nu_i > 0$ and

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \gamma^2 \neq 1, \xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}.$$

- $\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}$ with $\gamma^2 \neq 1$. We have

$$\tau(\xi) = -\frac{\gamma\,\mu_2 b}{\nu_1} X_1 + \frac{\gamma\, a}{\mu_2 \nu_1} X_2 + \frac{ba\,(\mu_2 - 1)}{\nu_2 \nu_1} X_3$$

and $\tau(\xi) = 0$ if and only if

$$(a \neq 0, b \neq 0, \gamma = 0, \mu_2 = 1)\,,\ ((a,b) \neq (0,0), ab = 0, \gamma = 0) \quad \text{or} \quad (a = b = 0).$$

- $\xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}$. We have

$$\tau(\xi) = -\frac{\mu_2 b}{\nu_1} X_1 + \frac{a}{\mu_2 \nu_1} X_2 + \frac{(\mu_2 - 1)(\alpha\beta\nu_1(\mu_1 - 1) + ab\mu_1)}{\mu_1 \nu_1 \nu_2} X_3$$

and $\tau(\xi) = 0$ if and only if

$$(a = b = 0, \alpha = 0), (a = b = 0, \beta = 0), (a = b = 0, \mu_1 = 1) \text{ or } (a = b = 0, \mu_2 = 1).$$

• $\xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}$. We have

$$\tau(\xi) = \frac{\mu_2 b}{\nu_1} X_1 - \frac{a}{\mu_2 \nu_1} X_2 + \frac{(\mu_2 - 1)(\alpha\beta\nu_1(\mu_1 - 1) + ab\mu_1)}{\mu_1 \nu_1 \nu_2} X_3$$

and $\tau(\xi) = 0$ if and only if

$$(a = b = 0, \alpha = 0), (a = b = 0, \beta = 0), (a = b = 0, \mu_1 = 1) \text{ or } (a = b = 0, \mu_2 = 1).$$

□

Theorem 3.2.2. *An homomorphism of* $e_0(2)$ *is biharmonic not harmonic if and only if it is conjugate to* $\xi : (e_0(2), \langle\ ,\ \rangle_1) \longrightarrow (e_0(2), \langle\ ,\ \rangle_2)$ *where* $\mathrm{Mat}(\langle\ ,\ \rangle_i, \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_i & 0 \\ 0 & 0 & \nu_i \end{pmatrix}, 0 < \mu_i \leq 1, \nu_i > 0$ *and either:*

1. $\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$ *and* $(a^2 = b^2, ab \neq 0)$,

2. $(\mu_1 \neq 0, \mu_2 \neq 1)$, $\xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}$ *and* $(a = b = 0, \alpha^2 = \beta^2, \alpha\beta \neq 0)$ *or*

$$\left(a = \epsilon b \mu_2 \sqrt{\mu_1}, b \neq 0, \alpha^2 = \beta^2 = \frac{\sqrt{\mu_1}\,({\mu_2}^2\nu_2 + a^2(\mu_2 - 1)^2)}{\mu_2\,\nu_1\,(\mu_2 - 1)^2\,(1 - \mu_1)}, \beta = \epsilon\alpha, \epsilon = \pm 1.\right)$$

3. $(\mu_1 \neq 0, \mu_2 \neq 1)$, $\xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}$ *and* $(a = b = 0, \alpha^2 = \beta^2, \alpha\beta \neq 0)$ *or*

$$\left(a = \epsilon b \mu_2 \sqrt{\mu_1}, b \neq 0, \alpha^2 = \beta^2 = \frac{\sqrt{\mu_1}\,({\mu_2}^2\nu_2 + a^2(\mu_2 - 1)^2)}{\mu_2\,\nu_1\,(\mu_2 - 1)^2\,(1 - \mu_1)}, \beta = \epsilon\alpha, \epsilon = \pm 1\right).$$

Proof. As in the proof of Theorem 3.2.1, according to Table 1.1, and homomorphism $\xi : (e_0(2), \langle\ ,\ \rangle_1) \longrightarrow (e_0(2), \langle\ ,\ \rangle_2)$ has, up to a conjugation, the form $\mathrm{Mat}(\langle\ ,\ \rangle_i, \mathbb{B}_0) = \mathrm{Diag}(1, \mu_i, \nu_i), i = 1, 2, 0 < \mu_i \leq 1, \nu_i > 0$ and

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \gamma^2 \neq 1, \xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}.$$

- $\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}$ with $\gamma^2 \neq 1$. We have

$$\begin{cases} \tau_2(\xi) = -\dfrac{b\gamma\left(\left(\gamma^2\nu_2 + a^2\right){\mu_2}^2 - 2\,a^2\mu_2 + a^2\right)}{{\nu_1}^2\nu_2}X_1 + \dfrac{\gamma\, a\,(b^2\mu_2(\mu_2-1)^2 + \gamma^2\nu_2)}{{\nu_1}^2{\mu_2}^2\nu_2}X_2 \\ +\dfrac{\left(\left(\gamma^2\nu_2 + a^2 - b^2\right){\mu_2}^2 + \left(\gamma^2\nu_2 - a^2 + b^2\right)\mu_2 + \gamma^2\nu_2\right)b\,(\mu_2 - 1)\,a}{{\nu_1}^2{\nu_2}^2\mu_2}X_3.. \end{cases}$$

If $\gamma = 0$ then

$$\tau_2(\xi) = \frac{(a-b)\,(a+b)\,(\mu_2-1)^2\,ba}{{\nu_1}^2{\nu_2}^2}X_3$$

and ξ is biharmonic not harmonic if and only if $a^2 = b^2$ and $ab \neq 0$.

If $\gamma \neq 0$ and $b \neq 0$ and $\tau_2(\xi) = 0$ then

$$\left(\gamma^2\nu_2 + a^2\right){\mu_2}^2 - 2\,a^2\mu_2 + a^2 = 0.$$

The discriminant of this equation on μ_2 is $\Delta = -4a^2\gamma^2\nu_2 \leq 0$ and this equation has no solution. It is also clear that if $\gamma \neq 0$ and $a \neq 0$ then $\tau_2(\xi) \neq 0$. In conclusion ξ is biharmonic not harmonic if and only if

$$\left(\gamma = 0, a^2 = b^2, ab \neq 0\right).$$

- $\xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}$. We have

$$\begin{cases} \tau(\xi) = -\dfrac{\mu_2 b}{\nu_1} X_1 + \dfrac{a}{\mu_2\nu_1} X_2 + \dfrac{(\mu_2-1)(\alpha\beta\nu_1(\mu_1-1)+ab\mu_1)}{\mu_1\nu_1\nu_2} X_3, \\ \tau_2(\xi) = A_1X_1 + A_2X_2 + A_3X_3, \\ A_1 = -\dfrac{\left(b\mu_1\left(\mu_2-1\right)^2 a^2 + \beta\,\alpha\,\nu_1\left(\mu_2-1\right)^2\left(\mu_1-1\right)a + b\mu_1{\mu_2}^2\nu_2\right)}{\mu_1{\nu_1}^2\nu_2}, \\ A_2 = \dfrac{\left(\left(\alpha\,\nu_1\left(\mu_1-1\right)\beta + ab\mu_1\right)b{\mu_2}^3 - 2\left(\alpha\,\nu_1\left(\mu_1-1\right)\beta+ab\mu_1\right)b{\mu_2}^2 + \left(\alpha\,\nu_1\left(\mu_1-1\right)\beta+ab\mu_1\right)b\mu_2 + a\mu_1\nu_2\right)}{{\nu_1}^2{\mu_2}^2\nu_2\mu_1} \\ {\mu_1}^2{\nu_1}^2{\nu_2}^2\mu_2A_3 = \mu_2\beta\,{\nu_1}^2\left(\mu_1-1\right)^2\left(\mu_2-1\right)^2\alpha^3 + \mu_2\nu_1ab\mu_1\left(\mu_2-1\right)^2\left(\mu_1-1\right)\alpha^2 \\ +\mu_2\beta\,\nu_1\left(\mu_1-1\right)\left(-\beta^2\mu_1\nu_1+a^2\mu_1-b^2\mu_1+\beta^2\nu_1\right)\left(\mu_2-1\right)^2\alpha + ab\mu_1\left(-\beta^2\mu_1{\mu_2}^2\nu_1+\mu_1{\mu_2}^2\nu_2+a^2\mu_1{\mu_2}^2\right. \\ \left.-b^2\mu_1{\mu_2}^2+\beta^2\mu_1\mu_2\nu_1+\beta^2{\mu_2}^2\nu_1+\mu_1\mu_2\nu_2-a^2\mu_1\mu_2+b^2\mu_1\mu_2-\beta^2\mu_2\nu_1+\mu_1\nu_2\right)\left(\mu_2-1\right) \end{cases}$$

and the test matrix is given by

$$M_\xi(\mathbb{B}_0) = \begin{pmatrix} \frac{\mu_2}{\nu_1} & 0 & -\frac{a\,(\mu_2-1)}{\nu_1} \\ 0 & \frac{1}{\nu_1} & \frac{b(\mu_2-1)}{\nu_1} \\ -\frac{\mu_2 a}{\nu_1} & -\frac{b}{\nu_1} & \frac{(\mu_2-1)\left(\left(\left(\alpha^2-\beta^2\right)\nu_1+a^2-b^2\right)\mu_1+\nu_1\left(-\alpha^2+\beta^2\right)\right)}{\mu_1\nu_1} \end{pmatrix} \quad \text{and} \quad \det(M_\xi(\mathbb{B})) = \frac{\mu_2(\mu_2-1)\left(\alpha^2-\beta^2\right)(\mu_1-1)}{{\nu_1}^2\mu_1}.$$

Note first that if $\mu_2 = 1$ then $\tau_2(\xi) = -\frac{b}{{\nu_1}^2}X_1 + \frac{a}{{\nu_1}^2}X_2$ and ξ is biharmonic if and only if it is harmonic. If $\mu_1 = 1$ then

$$A_1 = -\frac{\left(b\left(\mu_2-1\right)^2 a^2 + b{\mu_2}^2\nu_2\right)}{{\nu_1}^2\nu_2} \quad \text{and} \quad A_2 = \frac{\left(b^2a{\mu_2}^3 - 2\,b^2a{\mu_2}^2 + b^2a\mu_2 + a\nu_2\right)}{{\nu_1}^2{\mu_2}^2\nu_2}$$

and one can see easily $A_1 = A_2 = 0$ if and only if $a = b = 0$ and hence ξ is biharmonic if and only if ξ is biharmonic.

We suppose now that $\mu_1 < 1$ and $\mu_2 < 1$. So $\det(M_\xi(\mathbb{B}_0)) = 0$ if and only if $\alpha^2 = \beta^2$. According to Proposition 2.5.1, if $\alpha^2 \neq \beta^2$ then ξ is biharmonic if and only if it is harmonic. We have also that if $\alpha = \beta = 0$ then $\tau_2(\xi) = 0$ if and only if $a = b = 0$

Suppose that $\alpha^2 = \beta^2$ and $\alpha \neq 0$. If $a = b = 0$ then $\tau_2(\xi) = 0$ and ξ is biharmonic not harmonic. Suppose $(a, b) \neq 0$. Then the rank of $M_\xi(B_0)$ is equal to 2 and its kernel has dimension one and

$$v = a\left(\mu_2-1\right)X_1 - b\left(\mu_2-1\right)\mu_2X_2 + \mu_2X_3$$

is a generator of the kernel of $M_\xi(\mathbb{B}_0)$. But if ξ is biharmonic then $\tau(\xi)$ is in the kernel of $M_\xi(\mathbb{B}_0)$ and hence it is a multiple of v. Recall that

$$\tau(\xi) = -\frac{\mu_2 b}{\nu_1}X_1 + \frac{a}{\mu_2\nu_1}X_2 + \frac{(\mu_2-1)(\epsilon\alpha^2\nu_1(\mu_1-1)+ab\mu_1)}{\mu_1\nu_1\nu_2}X_3 \quad \text{and} \quad \epsilon = \pm 1.$$

But $(v, \tau(\xi))$ are linearly dependent if and only if

$$\begin{cases} \dfrac{(\mu_2-1)\,\nu_2\mu_1\,(-b^2{\mu_2}^3+a^2)}{\mu_2} = 0, \\ -b\,(\mu_2-1)^2\,\mu_2\epsilon\,(\mu_1-1)\,\nu_1\alpha^2 - a\mu_1\,(b^2{\mu_2}^3 - 2\,b^2{\mu_2}^2 + b^2\mu_2 + \nu_2) = 0, \\ b\mu_1\,(\mu_2-1)^2\,a^2 + \alpha^2\epsilon\,\nu_1\,(\mu_2-1)^2\,(\mu_1-1)\,a + b\mu_1{\mu_2}^2\nu_2 = 0. \end{cases}$$

Since $(a, b) \neq (0, 0)$, this is equivalent to

$$\begin{cases} a^2 = b^2{\mu_2}^3, \\ \alpha^2 = -\dfrac{\mu_1 a\,(b^2\mu_2(\mu_2-1)^2+\nu_2)}{b\,(\mu_2-1)^2\,\mu_2\epsilon\,(\mu_1-1)\,\nu_1} = -\dfrac{\mu_1 b\,({\mu_2}^2\nu_2 + a^2(\mu_2-1)^2)}{\epsilon\,\nu_1\,(\mu_2-1)^2\,(\mu_1-1)\,a} \end{cases}$$

and this is equivalent to

$$\begin{cases} a^2 = b^2{\mu_2}^3, \\ \alpha^2 = -\dfrac{\mu_1 b\,({\mu_2}^2\nu_2 + a^2(\mu_2-1)^2)}{\epsilon\,\nu_1\,(\mu_2-1)^2\,(\mu_1-1)\,a}. \end{cases}$$

So $a = \epsilon b\mu_2\sqrt{\mu_2}$ and we get the desired result.

The case of $\xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}$ can be treated identically. □

3.3 Harmonic and biharmonic homomorphisms on Sol

Theorem 3.3.1. *An homomorphism of* $\mathfrak{sol}$ *is harmonic if and only if it is conjugate to* $\xi : (\mathfrak{sol}, \langle\ ,\ \rangle_1) \longrightarrow (\mathfrak{sol}, \langle\ ,\ \rangle_2)$ *where:*

1. $\langle\ ,\ \rangle_i = \mathrm{Diag}(1, 1, \nu_i)$, $i = 1, 2$ *and* $\nu_i > 0$ *and either*

$$[\xi = \xi_1,\ (a = b = 0) \quad or \quad (\gamma = 0, a^2 = b^2)], [\xi = \xi_2,\ (a = b = 0, \alpha^2 = \beta^2)]$$
$$or \quad [\xi = \xi_3,\ (a = b = 0, \alpha^2 = \beta^2)].$$

2. $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}, \langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ *and* $\nu_i > 0,\ \mu_2 > 1$ *and either*

$$\left[\xi = \xi_1, (a = b = 0) \quad or \quad \left(\gamma = 0, \mu_2 = \frac{a^2}{b^2}\right)\right],$$
$$\left[\xi = \xi_2, (a = b = \alpha = \beta = 0) \quad or \quad \left(a = b = 0, \mu_2 = \frac{\alpha^2}{\beta^2}\right)\right] \quad or$$
$$\left[\xi = \xi_3, (a = b = \alpha = \beta = 0) \quad or \quad \left(a = b = 0, \mu_2 = \frac{\alpha^2}{\beta^2}\right)\right].$$

3. $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}, \langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ *and* $\nu_i > 0,\ \mu_1 > 1,$

$$\left[\xi = \xi_1, (a = b = 0) \quad or \quad (\gamma = 0, a^2 = b^2)\right],$$
$$\left[\xi = \xi_2, (a = b = \alpha = \beta = 0) \quad or \quad \left(a = b = 0, \mu_1 = \frac{\beta^2}{\alpha^2}\right)\right], \quad or$$
$$\left[\xi = \xi_3, (a = b = \alpha = \beta = 0) \quad or \quad \left(a = b = 0, \mu_1 = \frac{\beta^2}{\alpha^2}\right)\right].$$

4. $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}, \langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ *and* $\nu_i > 0,\ \mu_i > 1$ *and either*

$$\left[\xi = \xi_1, (a = b = 0) \quad or \quad \left(\gamma = 0, \mu_2 = \frac{a^2}{b^2}\right)\right],$$
$$\left[\xi = \xi_2, (a = b = \alpha = \beta = 0) \quad or \quad \left(a = b = 0, (\alpha, \beta) \neq (0,0), \alpha^2\mu_1 = \beta^2\mu_2\right)\right] \quad or$$
$$\left[\xi = \xi_3, (a = b = \alpha = \beta = 0) \quad or \quad \left(a = b = 0, (\alpha, \beta) \neq (0,0), \alpha^2\mu_1\mu_2 = \beta^2\right)\right].$$

The homomorphisms $\xi_i, i = 1..3$ *are given by*

$$\xi_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \ \xi_2 = \begin{pmatrix} \alpha & 0 & a \\ 0 & \beta & b \\ 0 & 0 & 1 \end{pmatrix} \quad and \quad \xi_3 = \begin{pmatrix} 0 & \beta & a \\ \alpha & 0 & b \\ 0 & 0 & -1 \end{pmatrix}.$$

Proof. We use Table 1.1 to get all the conjugation classes of homomorphisms of $\mathfrak{sol}$ and for each one we compute $\tau(\xi)$.

- $\langle\ ,\ \rangle_i = \mathrm{Diag}(1,1,\nu_i)$, $i=1,2$ and $\nu_i>0$. We have

$$\begin{cases}\tau(\xi_1) = -\dfrac{\gamma\, a}{\nu_1}X_1 + \dfrac{\gamma\, b}{\nu_1}X_2 + \dfrac{a^2-b^2}{\nu_2\nu_1}X_3,\\ \tau(\xi_2) = -\dfrac{a}{\nu_1}X_1 + \dfrac{b}{\nu_1}X_2 + \dfrac{(\alpha^2-\beta^2)\,\nu_1 + a^2 - b^2}{\nu_2\nu_1}X_3,\\ \tau(\xi_3) = \dfrac{a}{\nu_1}X_1 - \dfrac{b}{\nu_1}X_2 + \dfrac{(-\alpha^2+\beta^2)\,\nu_1 + a^2 - b^2}{\nu_2\nu_1}X_3.\end{cases}$$

- $\langle\ ,\ \rangle_1 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \nu_1\end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix}1 & 1 & 0\\ 1 & \mu_2 & 0\\ 0 & 0 & \nu_2\end{pmatrix}$ and $\nu_i>0$, $\mu_2>1$. We have

$$\begin{cases}\tau(\xi_1) = -\dfrac{((a+2\,b)\,\mu_2 + a)\,\gamma}{(\mu_2-1)\,\nu_1}X_1 + \dfrac{\gamma\,(b\mu_2 + 2\,a + b)}{(\mu_2-1)\,\nu_1}X_2 + \dfrac{-b^2\mu_2 + a^2}{\nu_2\nu_1}X_3,\\ \tau(\xi_2) = -\dfrac{(a+2\,b)\,\mu_2 + a}{(\mu_2-1)\,\nu_1}X_1 + \dfrac{b\mu_2 + 2\,a + b}{(\mu_2-1)\,\nu_1}X_2 + \dfrac{(-\beta^2\mu_2+\alpha^2)\,\nu_1 - b^2\mu_2 + a^2}{\nu_2\nu_1}X_3,\\ \\ \tau(\xi_3) = \dfrac{(a+2\,b)\,\mu_2 + a}{(\mu_2-1)\,\nu_1}X_1 - \dfrac{\mu_2 b + 2\,a + b}{(\mu_2-1)\,\nu_1}X_2 + \dfrac{(-\alpha^2\mu_2+\beta^2)\,\nu_1 - b^2\mu_2 + a^2}{\nu_2\nu_1}X_3.\end{cases}$$

- $\langle\ ,\ \rangle_1 = \begin{pmatrix}1 & 1 & 0\\ 1 & \mu_1 & 0\\ 0 & 0 & \nu_1\end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \nu_2\end{pmatrix}$ and $\nu_i>0$, $\mu_1>1$. We have

$$\begin{cases}\tau(\xi_1) = -\dfrac{\gamma\, a}{\nu_1}X_1 + \dfrac{\gamma\, b}{\nu_1}X_2 + \dfrac{a^2-b^2}{\nu_2\nu_1}X_3,\\ \tau(\xi_2) = -\dfrac{a}{\nu_1}X_1 + \dfrac{b}{\nu_1}X_2 + \dfrac{(\alpha^2\nu_1 + a^2 - b^2)\,\mu_1 - \beta^2\nu_1 - a^2 + b^2}{\nu_2\,(\mu_1-1)\,\nu_1}X_3,\\ \tau(\xi_3) = \dfrac{a}{\nu_1}X_1 - \dfrac{b}{\nu_1}X_2 + \dfrac{(-\alpha^2\nu_1 + a^2 - b^2)\,\mu_1 + \beta^2\nu_1 - a^2 + b^2}{\nu_2\,(\mu_1-1)\,\nu_1}X_3.\end{cases}$$

• $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_i > 1$. We have

$$\begin{cases} \tau(\xi_1) = -\dfrac{((a+2\,b)\,\mu_2 + a)\,\gamma}{(\mu_2 - 1)\,\nu_1} X_1 + \dfrac{\gamma\,(b\mu_2 + 2\,a + b)}{(\mu_2 - 1)\,\nu_1} X_2 + \dfrac{-b^2\mu_2 + a^2}{\nu_2\nu_1} X_3, \\ \tau(\xi_2) = -\dfrac{(a+2\,b)\,\mu_2 + a}{(\mu_2 - 1)\,\nu_1} X_1 + \dfrac{b\mu_2 + 2\,a + b}{(\mu_2 - 1)\,\nu_1} X_2 + \dfrac{(\alpha^2\nu_1 - b^2\mu_2 + a^2)\,\mu_1 + (-\beta^2\nu_1 + b^2)\,\mu_2 - a^2}{\nu_2\,(\mu_1 - 1)\,\nu_1} X_3, \\ \tau(\xi_3) = \dfrac{(a+2\,b)\,\mu_2 + a}{(\mu_2 - 1)\,\nu_1} X_1 - \dfrac{b\mu_2 + 2\,a + b}{(\mu_2 - 1)\,\nu_1} X_2 + \dfrac{((-\alpha^2\nu_1 - b^2)\,\mu_2 + a^2)\,\mu_1 + b^2\mu_2 + \beta^2\nu_1 - a^2}{\nu_2\,(\mu_1 - 1)\,\nu_1} X_3. \end{cases}$$

One can check that $\tau(\xi_i) = 0$ are equivalent to the conditions given in the theorem. □

Theorem 3.3.2. *An homomorphism of $\mathfrak{sol}$ is biharmonic if and only if it is harmonic.*

Proof. As above, we put

$$\xi_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix},\ \xi_2 = \begin{pmatrix} \alpha & 0 & a \\ 0 & \beta & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \xi_3 = \begin{pmatrix} 0 & \beta & a \\ \alpha & 0 & b \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\xi : (\mathfrak{sol}, \langle\ ,\ \rangle_1) \longrightarrow (\mathfrak{sol}, \langle\ ,\ \rangle_2)$ an homomorphism. Table 1.1 gives all the possible conjugation classes of ξ and we will show that for each case ξ is biharmonic if and only if ξ is harmonic.

• $\xi = \xi_1$ and $\langle\ ,\ \rangle_i = \mathrm{Diag}(1, 1, \nu_i)$, $i = 1, 2$ and $\nu_i > 0$. We have

$$\tau_2(\xi) = -2\,\frac{(1/2\,\gamma^2\nu_2 + a^2 - b^2)\,a\gamma}{{\nu_1}^2\nu_2} X_1 - 2\,\frac{(-1/2\,\gamma^2\nu_2 + a^2 - b^2)\,\gamma\,b}{{\nu_1}^2\nu_2} X_2 + \frac{\gamma^2\,(a^2 - b^2)\,\nu_2 + 2\,a^4 - 2\,b^4}{{\nu_2}^2{\nu_1}^2} X_3.$$

One can see easily that $\tau_2(\xi) = 0$ if and only if $(a = b = 0)$ or $(\gamma = 0, a^2 = b^2)$ which, according to Theorem 3.3.1, is equivalent to ξ is harmonic.

• $\xi = \xi_2$ and $\langle\ ,\ \rangle_i = \mathrm{Diag}(1, 1, \nu_i)$, $i = 1, 2$ and $\nu_i > 0$. We have

$$\begin{cases} \tau_2(\xi) = -2\,\dfrac{a\,((\alpha^2 - \beta^2)\,\nu_1 + a^2 - b^2 + 1/2\,\nu_2)}{{\nu_1}^2\nu_2} X_1 - 2\,\dfrac{((\alpha^2 - \beta^2)\,\nu_1 + a^2 - b^2 - 1/2\,\nu_2)\,b}{{\nu_1}^2\nu_2} X_2 \\ + \dfrac{2\,\alpha^4{\nu_1}^2 - 2\,\beta^4{\nu_1}^2 + 4\,a^2\alpha^2\nu_1 - 4\,b^2\beta^2\nu_1 + 2\,a^4 - 2\,b^4 + a^2\nu_2 - b^2\nu_2}{{\nu_1}^2{\nu_2}^2} X_3. \end{cases}$$

We have also

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} {\nu_1}^{-1} & 0 & -2\,\frac{a}{\nu_1} \\ 0 & {\nu_1}^{-1} & -2\,\frac{b}{\nu_1} \\ -\frac{a}{\nu_1} & -\frac{b}{\nu_1} & \frac{2\,\alpha^2\nu_1 + 2\,\beta^2\nu_1 + 2\,a^2 + 2\,b^2}{\nu_1} \end{bmatrix} \quad \text{and} \quad \det(M_\xi(\mathbb{B}_0)) = 2\,\frac{\alpha^2 + \beta^2}{{\nu_1}^2}.$$

According to Proposition 2.5.1, if $(\alpha, \beta) \neq (0,0)$ then ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_3$ and $\langle\ ,\ \rangle_i = \mathrm{Diag}(1,1,\nu_i)$, $i = 1,2$ and $\nu_i > 0$. We have

$$\begin{cases} \tau_2(\xi) = \dfrac{a\,(-2\,\alpha^2\nu_1 + 2\,\beta^2\nu_1 + 2\,a^2 - 2\,b^2 + \nu_2)}{{\nu_1}^2\nu_2}X_1 + \dfrac{b\,(-2\,\alpha^2\nu_1 + 2\,\beta^2\nu_1 + 2\,a^2 - 2\,b^2 - \nu_2)}{{\nu_1}^2\nu_2}X_2 \\ + \dfrac{-2\,\alpha^4{\nu_1}^2 + 2\,\beta^4{\nu_1}^2 + 4\,a^2\beta^2\nu_1 - 4\,\alpha^2b^2\nu_1 + 2\,a^4 - 2\,b^4 + a^2\nu_2 - b^2\nu_2}{{\nu_1}^2{\nu_2}^2}X_3 \end{cases}$$

and

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} {\nu_1}^{-1} & 0 & 2\,\frac{a}{\nu_1} \\ 0 & {\nu_1}^{-1} & 2\,\frac{b}{\nu_1} \\ \frac{a}{\nu_1} & \frac{b}{\nu_1} & \frac{2\,\alpha^2\nu_1 + 2\,\beta^2\nu_1 + 2\,a^2 + 2\,b^2}{\nu_1} \end{bmatrix} \quad \text{and} \quad \det(M_\xi(\mathbb{B}_0)) = 2\,\frac{\alpha^2+\beta^2}{{\nu_1}^2}$$

and the situation is similar to the precedent cases.

- $\xi = \xi_1$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_2 > 1$.

$$\begin{cases} \tau(\xi) = -\dfrac{\gamma\ ((a+2\,b)\,\mu_2 + a)}{(\mu_2 - 1)\,\nu_1}X_1 + \dfrac{\gamma\ (b\mu_2 + 2\,a + b)}{(\mu_2-1)\,\nu_1}X_2 + \dfrac{-b^2\mu_2 + a^2}{\nu_2\nu_1}X_3, \\ \tau_2(\xi) = -2\,\dfrac{(-b^2\,(a-b)\,{\mu_2}^3 + (a^3 - a^2b + (1/2\,\gamma^2\nu_2 + b^2)\,a + 2\,b\gamma^2\nu_2 - b^3)\,{\mu_2}^2 + (3\,a\gamma^2\nu_2 + 2\,b\gamma^2\nu_2 - a^3 + a^2}{{\nu_1}^2\nu_2\,(\mu_2-1)^2} \\ +2\,\dfrac{\gamma\ (b^3{\mu_2}^3 - b\,(-1/2\,\gamma^2\nu_2 + a^2 + ab + b^2)\,{\mu_2}^2 + (ab^2 + (3\,\gamma^2\nu_2 + a^2)\,b + 2\,a\gamma^2\nu_2 + a^3)\,\mu_2 + 2\,a\gamma^2\nu_2 + 1/2\,b\gamma}{{\nu_1}^2\nu_2\,(\mu_2-1)^2} \\ +2\,\dfrac{(-b^2\mu_2 + a^2)\,(b^2{\mu_2}^2 + (1/2\,\gamma^2\nu_2 + a^2 - b^2)\,\mu_2 + 3/2\,\gamma^2\nu_2 - a^2)}{{\nu_1}^2{\nu_2}^2\,(\mu_2-1)}X_3 \end{cases}$$

Suppose that ξ is biharmonic not harmonic. Then, by virtue of Theorem 3.3.1, $(a,b) \neq 0$ and ($\gamma \neq 0$ or $\mu_2 \neq \frac{a^2}{b^2}$). If $\mu_2 = \frac{a^2}{b^2}$ then a direct computation shows that

$$\tau_2(\xi) = -\frac{(a+b)^2\,\gamma^3 a}{(a-b)^2\,{\nu_1}^2}X_1 + \frac{b\,(a+b)^2\,\gamma^3}{(a-b)^2\,{\nu_1}^2}X_2$$

and since $(a,b) \neq (0,0)$, $\gamma \neq 0$ and $\mu_2 > 1$ this is impossible so we must have $\mu_2 \neq \frac{a^2}{b^2}$. In this case, since the last coordinate of $\tau_2(\xi)$ vanishes, we get

$$\left(\frac{1}{2}\,\gamma^2\nu_2 + a^2\right)\mu_2 + \frac{3}{2}\,\gamma^2\nu_2 - a^2 + \left({\mu_2}^2 - \mu_2\right)b^2 = \frac{1}{2}\,\gamma^2\nu_2\mu_2 + a^2(\mu_2 - 1) + \frac{3}{2}\,\gamma^2\nu_2 + \left({\mu_2}^2 - \mu_2\right)b^2 = 0$$

But since $\mu_2 > 1$, this is equivalent to $\gamma = a = b = 0$ which is a contradiction. Finally, ξ is biharmonic if and only if it is harmonic.

- $\xi = \xi_2$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_2 > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} {\nu_1}^{-1} & -{\nu_1}^{-1} & -2\,\frac{a}{\nu_1} \\ -{\nu_1}^{-1} & \frac{\mu_2}{\nu_1} & -2\,\frac{b\mu_2}{\nu_1} \\ \frac{-a+b}{\nu_1} & \frac{-b\mu_2+a}{\nu_1} & \frac{2\,\nu_1\beta^2\mu_2+2\,\alpha^2\nu_1+2\,b^2\mu_2+2\,a^2}{\nu_1} \end{bmatrix} \quad \text{and} \quad \det(M_\xi(\mathbb{B}_0)) = 2\,\frac{(\mu_2-1)\,(\beta^2\mu_2+\alpha^2)}{{\nu_1}^2}$$

If $(\alpha, \beta) \neq (0,0)$ then, according to Proposition 2.5.1, ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_3$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_2 > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} {\nu_1}^{-1} & -{\nu_1}^{-1} & 2\,\frac{a}{\nu_1} \\ -{\nu_1}^{-1} & \frac{\mu_2}{\nu_1} & 2\,\frac{b\mu_2}{\nu_1} \\ \frac{a-b}{\nu_1} & \frac{b\mu_2-a}{\nu_1} & \frac{2\,\alpha^2\mu_2\nu_1+2\,b^2\mu_2+2\,\beta^2\nu_1+2\,a^2}{\nu_1} \end{bmatrix} \quad \text{and} \quad \det(M_\xi(\mathbb{B}_0)) = 2\,\frac{(\mu_2-1)\,(\alpha^2\mu_2+\beta^2)}{{\nu_1}^2}$$

If $(\alpha, \beta) \neq (0,0)$ then, according to Proposition 2.5.1, ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_1$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_1 > 1$. We have

$$\tau_2(\xi) = -2\,\frac{\gamma\,(1/2\,\gamma^2\nu_2+a^2-b^2)\,a}{{\nu_1}^2\nu_2}X_1 - 2\,\frac{b\,(-1/2\,\gamma^2\nu_2+a^2-b^2)\,\gamma}{{\nu_1}^2\nu_2}X_2 + \frac{\gamma^2\,(a^2-b^2)\,\nu_2+2\,a^4-2\,b^4}{{\nu_2}^2{\nu_1}^2}X_3$$

One can see easily that $\tau_2(\xi) = 0$ if and only if $(a = b = 0)$ or $(\gamma = 0, a^2 = b^2)$ which, according to Theorem 3.3.1, is equivalent to ξ is harmonic.

- $\xi = \xi_2$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_1 > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} {\nu_1}^{-1} & 0 & -2\,\frac{a}{\nu_1} \\ 0 & {\nu_1}^{-1} & -2\,\frac{b}{\nu_1} \\ -\frac{a}{\nu_1} & -\frac{b}{\nu_1} & \frac{\left(2\,\alpha^2\nu_1+2\,a^2+2\,b^2\right)\mu_1+2\,\beta^2\nu_1-2\,a^2-2\,b^2}{(\mu_1-1)\nu_1} \end{bmatrix} \quad \text{and} \quad \det(M_\xi(\mathbb{B}_0)) = 2\,\frac{\alpha^2\mu_1+\beta^2}{{\nu_1}^2\,(\mu_1-1)}.$$

If $(\alpha,\beta) \neq (0,0)$ then, according to Proposition 2.5.1, ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_3$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_1 > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} {\nu_1}^{-1} & 0 & 2\,\frac{a}{\nu_1} \\ 0 & {\nu_1}^{-1} & 2\,\frac{b}{\nu_1} \\ \frac{a}{\nu_1} & \frac{b}{\nu_1} & \frac{\left(2\,\alpha^2\nu_1+2\,a^2+2\,b^2\right)\mu_1+2\,\beta^2\nu_1-2\,a^2-2\,b^2}{(\mu_1-1)\nu_1} \end{bmatrix} \quad \text{and} \quad \det(M_\xi(\mathbb{B}_0)) = 2\,\frac{\alpha^2\mu_1+\beta^2}{{\nu_1}^2\,(\mu_1-1)}.$$

The situation is similar to the precedent case.

- $\xi = \xi_1$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_i > 1$. We have

$$\begin{cases} \tau_2(\xi) = -2\,\dfrac{\gamma\,\left(-b^2\,(a-b)\,{\mu_2}^3 + \left(a^3 - a^2b + \left(1/2\,\gamma^2\nu_2 + b^2\right)a + 2\,b\gamma^2\nu_2 - b^3\right){\mu_2}^2 + \left(3\,a\gamma^2\nu_2 + 2\,b\gamma^2\nu_2 - a^3 + \right.\right.}{{\nu_1}^2\nu_2\,(\mu_2-1)^2} \\ +2\,\dfrac{\gamma\,\left(b^3{\mu_2}^3 - b\left(-1/2\,\gamma^2\nu_2 + a^2 + ab + b^2\right){\mu_2}^2 + \left(ab^2 + \left(3\,\gamma^2\nu_2 + a^2\right)b + 2\,a\gamma^2\nu_2 + a^3\right)\mu_2 + 2\,a\gamma^2\nu_2 + 1/2\,b\gamma^2\right.}{{\nu_1}^2\nu_2\,(\mu_2-1)^2} \\ 2\,\dfrac{\left(-b^2\mu_2 + a^2\right)\left(b^2{\mu_2}^2 + \left(1/2\,\gamma^2\nu_2 + a^2 - b^2\right)\mu_2 + 3/2\,\gamma^2\nu_2 - a^2\right)}{{\nu_1}^2{\nu_2}^2\,(\mu_2-1)}X_3 \end{cases}$$

One can see that $\tau_2(\xi)$ is the same as in the case $\xi = \xi_1$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and we can use the same arguments to conclude.

- $\xi = \xi_2$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_i > 1$. We have

$$M_\xi(\mathbb{B}_0) \begin{bmatrix} {\nu_1}^{-1} & -{\nu_1}^{-1} & -2\,\frac{a}{\nu_1} \\ -{\nu_1}^{-1} & \frac{\mu_2}{\nu_1} & -2\,\frac{\mu_2 b}{\nu_1} \\ \frac{-a+b}{\nu_1} & \frac{-\mu_2 b+a}{\nu_1} & \frac{\left(2\,\alpha^2\nu_1+2\,b^2\mu_2+2\,a^2\right)\mu_1+\left(2\,\beta^2\nu_1-2\,b^2\right)\mu_2-2\,a^2}{\nu_1(\mu_1-1)} \end{bmatrix} \text{ and}$$

$$\det(M_\xi(\mathbb{B}_0)) = 2\,\frac{(\mu_2-1)\,(\alpha^2\mu_1+\beta^2\mu_2)}{{\nu_1}^2\,(\mu_1-1)}.$$

If $(\alpha,\beta) \neq (0,0)$ then, according to Proposition 2.5.1, ξ is biharmonic if and only if it is harmonic. If $\alpha=\beta=0$ then $\xi=\xi_1$ with $\gamma=1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_3$ and $\langle\ ,\ \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle\ ,\ \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_i > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} {\nu_1}^{-1} & -{\nu_1}^{-1} & 2\,\frac{a}{\nu_1} \\ -{\nu_1}^{-1} & \frac{\mu_2}{\nu_1} & 2\,\frac{\mu_2 b}{\nu_1} \\ \frac{a-b}{\nu_1} & \frac{\mu_2 b-a}{\nu_1} & \frac{\left(\left(2\,\alpha^2\nu_1+2\,b^2\right)\mu_2+2\,a^2\right)\mu_1-2\,b^2\mu_2+2\,\beta^2\nu_1-2\,a^2}{\nu_1(\mu_1-1)} \end{bmatrix} \text{ and}$$

$$\det(M_\xi(B_0)) = 2\,\frac{(\mu_2-1)\,(\alpha^2\mu_1\mu_2+\beta^2)}{{\nu_1}^2\,(\mu_1-1)}.$$

The situation is similar to the precedent case. □

3.4 Harmonic and biharmonic homomorphisms of $\mathfrak{su}(2)$

The following proposition is a consequence of [3, Proposition 2.5].

Proposition 3.4.1. *Let $\xi : (\mathfrak{su}(2), \langle\ ,\ \rangle_1) \longrightarrow (\mathfrak{su}(2), \langle\ ,\ \rangle_2)$ be an automorphism. If $\langle\ ,\ \rangle_1$ or $\langle\ ,\ \rangle_2$ is bi-invariant then ξ is harmonic.*

Any homomorphism of su(2) is an automorphism and it is a product $\xi_3(a) \circ \xi_2(b) \circ \xi_1(c)$ where

$$\xi_1(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(a) & \sin(a) \\ 0 & -\sin(a) & \cos(a) \end{pmatrix}, \xi_2(a) = \begin{pmatrix} \cos(a) & 0 & \sin(a) \\ 0 & 1 & 0 \\ -\sin(a) & 0 & \cos(a) \end{pmatrix}, \xi_3(a) = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ -\sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\xi_i : (\mathfrak{su}(2), \langle\ ,\ \rangle_1) \longrightarrow (\mathfrak{su}(2), \langle\ ,\ \rangle_2)$ with $\langle\ ,\ \rangle_j = \mathrm{Diag}(\lambda_j, \mu_j, \nu_j)$ then

$$\begin{cases} \tau(\xi_1(a)) = -\dfrac{\sin(a)\cos(a)(\mu_2-\nu_2)(\mu_1-\nu_1)}{\lambda_2\mu_1\nu_1}X_1, \\ \tau_2(\xi_1(a)) = -2\,\dfrac{(\mu_2-\nu_2)^2(-\nu_1+\mu_1)^2\cos(a)\left((\cos(a))^2-1/2\right)\sin(a)}{\mu_1{}^2\nu_1{}^2\lambda_2{}^2}X_1, \\ \tau(\xi_2(a)) = \dfrac{\sin(a)\cos(a)(\lambda_2-\nu_2)(\lambda_1-\nu_1)}{\mu_2\lambda_1\nu_1}X_2, \\ \tau_2(\xi_2(a)) = \dfrac{\left(2\,(\cos(a))^2-1\right)\cos(a)(\lambda_2-\nu_2)^2(-\nu_1+\lambda_1)^2\sin(a)}{\lambda_1{}^2\nu_1{}^2\mu_2{}^2}X_2, \\ \tau(\xi_3(a)) = -\dfrac{\cos(a)\sin(a)(\lambda_2-\mu_2)(\lambda_1-\mu_1)}{\lambda_1\mu_1\nu_2}X_3, \\ \tau_2(\xi_3(a)) = -2\,\dfrac{\cos(a)\sin(a)\left((\cos(a))^2-1/2\right)(\mu_2-\lambda_2)^2(\mu_1-\lambda_1)^2}{\lambda_1{}^2\mu_1{}^2\nu_2{}^2}X_3. \end{cases}$$

So we get:

Proposition 3.4.2. *1. If $\mu_2 = \nu_2$ or $\mu_1 = \nu_1$ then $\xi_1(a)$ is harmonic.*

2. *If $\mu_2 \neq \nu_2$ and $\mu_1 \neq \nu_1$ then $\xi_1(a)$ is harmonic if and only if $\sin(2a) = 0$ and $\xi_1(a)$ is biharmonic not harmonic if and only if $\cos(a)^2 = \frac{1}{2}$.*

3. *If $\lambda_2 = \nu_2$ or $\lambda_1 = \nu_1$ then $\xi_2(a)$ is harmonic.*

4. *If $\lambda_2 \neq \nu_2$ and $\lambda_1 \neq \nu_1$ then $\xi_2(a)$ is harmonic if and only if $\sin(2a) = 0$ and $\xi_2(a)$ is biharmonic not harmonic if and only if $\cos(a)^2 = \frac{1}{2}$.*

5. *If $\lambda_2 = \mu_2$ or $\lambda_1 = \mu_1$ then $\xi_3(a)$ is harmonic.*

6. *If $\lambda_2 \neq \mu_2$ and $\lambda_1 \neq \mu_1$ then $\xi_3(a)$ is harmonic if and only if $\sin(2a)$ and $\xi_3(a)$ is biharmonic not harmonic if and only if $\cos(a)^2 = \frac{1}{2}$.*

Theorem 3.4.1. *We consider the automorphism*

$$\xi = \xi_3(a)\circ\xi_2(b)\circ\xi_1(c) : (\mathfrak{su}(2), \mathrm{diag}(\lambda_1, \mu_1, \nu_1)) \longrightarrow (\mathfrak{su}(2), \mathrm{diag}(\lambda_2, \mu_2, \nu_2))\,,\ 0 \leq \nu_i < \mu_i \leq \lambda_i, i = 1, 2.$$

1. *If $(0 < \nu_1 < \mu_1 < \lambda_1, 0 < \nu_2 < \mu_2 < \lambda_2)$ or $(0 < \nu_1 < \mu_1 < \lambda_1, 0 < \nu_2 < \mu_2 < \lambda_2)$ then ξ is harmonic if and only if one of the following condition holds:*

   (i) $\cos(b) = 0$, $\sin(b) = 1$ *and* $\sin(2(a-c)) = 0$,

   (ii) $\cos(b) = 0$, $\sin(b) = -1$ *and* $\sin(2(a+c)) = 0$, (*iii*) $\sin(b) = 0$ *and* $\sin(2c) = \sin(2a) = 0$.

2. *If* $(0 < \nu_1 < \mu_1 < \lambda_1, 0 < \nu_2 = \mu_2 < \lambda_2)$ *then* ξ *is harmonic if and only if one of the following condition holds:*

   (*i*) $\cos(b) = 0$, $\sin(b) = 1$ *and* $\sin(2(a-c)) = 0$,

   (*ii*) $\cos(b) = 0$, $\sin(b) = -1$ *and* $\sin(2(a+c)) = 0$,

   (*iii*) $\sin(b) = 0$ *and* $\sin(a) = 0$.

   (*iv*) $\sin(b) = 0$, $\sin(2c) = 0$ *and* $\cos(a) = 0$.

3. *If* $(0 < \nu_1 < \mu_1 = \lambda_1, 0 < \nu_2 < \mu_2 < \lambda_2)$ *then* ξ *is harmonic if and only if one of the following condition holds:*

   (*i*) $\cos(b) = 0$, $\sin(b) = 1$ *and* $\sin(2(a-c)) = 0$,

   (*ii*) $\cos(b) = 0$, $\sin(b) = -1$ *and* $\sin(2(a+c)) = 0$,

   (*iii*) $\sin(b) = 0$ *and* $\sin(a) = \sin(2c) = 0$.

   (*iv*) $\sin(b) = 0$ *and* $\cos(a) = \sin(2c) = 0$.

4. *If* $(0 < \nu_1 < \mu_1 < \lambda_1, 0 < \nu_2 < \mu_2 = \lambda_2)$ *then* ξ *is harmonic if and only if* $\cos(b) = 0$ *or* $(\sin(b) = \sin(2c) = 0)$.

5. *If* $(0 < \nu_1 = \mu_1 < \lambda_1, 0 < \nu_2 < \mu_2 < \lambda_2)$ *then* ξ *is harmonic if and only if* $\cos(b) = 0$ *or* $(\sin(b) = \sin(2a) = 0)$.

6. *If* $(0 < \nu_1 = \mu_1 < \lambda_1, 0 < \nu_2 = \mu_2 < \lambda_2)$ *then* ξ *is harmonic if and only if* $(\cos(b) = 0)$, $(\cos(a)) = 0$ *or* $(\sin(b) = \sin(a) = 0)$.

7. *If* $(0 < \nu_1 = \mu_1 < \lambda_1, 0 < \nu_2 < \mu_2 = \lambda_2)$ *then* ξ *is harmonic if and only if* $\sin(2b) = 0$.

8. *If* $(0 < \nu_1 < \mu_1 = \lambda_1, 0 < \nu_2 < \mu_2 = \lambda_2)$ *then* ξ *is harmonic if and only if* $\cos(b)\cos(c) = 0$ *or* $(\sin(b) = \sin(c) = 0)$.

9. *If* $(0 < \nu_1 < \mu_1 = \lambda_1, 0 < \nu_2 = \mu_2 < \lambda_2)$ *then* ξ *is harmonic if and only if one of the following situations holds*

   (*i*) $\cos(b) = 0$, $\sin(b) = 1$ *and* $\sin(2(a-c)) = 0$,

   (*ii*) $\cos(b) = 0$, $\sin(b) = -1$ *and* $\sin(2(a+c)) = 0$, (*iii*) $\cos(c) = 0$ *and* $\sin(2a) = 0$,

(*iv*) $\sin(b) = \sin(c) = 0$,

(*v*) $\cos(a) = (-1)^k \frac{\sin(c)}{\sqrt{\sin^2(c)+\sin^2(b)\cos^2(c)}}$ *and* $\sin(a) = (-1)^{k+1} \frac{\sin(b)\cos(c)}{\sqrt{\sin^2(c)+\sin^2(b)\cos^2(c)}}$.

Proof. We have

$$\begin{cases} \tau(\xi) = A_1X_1 + A_2X_2 + A_3X_3, \\ \lambda_2\lambda_1\mu_1\nu_1A_1 = \cos(b)\Big(\sin(a)\sin(b)\,\lambda_1(\mu_1-\nu_1)(\cos(c))^2 - \sin(c)\,\lambda_1\cos(a)(\mu_1-\nu_1)\cos(c) + \sin(a)\sin(b)\,\nu_1(\lambda_1 - \\ = \cos(b)(\mu_2-\nu_2)\,R, \\ \mu_2\lambda_1\mu_1\nu_1A_2 = \cos(b)\Big(\sin(b)\Big(\lambda_1(\mu_1-\nu_1)(\cos(c))^2 + \nu_1(\lambda_1-\mu_1)\Big)\cos(a) + \cos(c)\sin(a)\sin(c)\,\lambda_1(\mu_1-\nu_1)\Big)(\lambda_2 - \\ = (\lambda_2-\nu_2)\cos(b)S, \\ z := -\nu_2\lambda_1\mu_1\nu_1A_3 = (\lambda_2-\mu_2)\Big(2\cos(c)\sin(b)\sin(c)\,\lambda_1(\mu_1-\nu_1)(\cos(a))^2 \\ +\Big(\lambda_1\Big((\cos(b))^2-2\Big)(\mu_1-\nu_1)(\cos(c))^2 + \nu_1(\lambda_1-\mu_1)(\cos(b))^2 + \lambda_1(\mu_1-\nu_1)\Big)\sin(a)\cos(a) - \cos(c)\sin(b)\sin(c \end{cases}$$

On the other hand, the following relations are straightforward to establish:

$$\begin{cases} R\cos(a) - S\sin(a) = -\lambda_1(\mu_1-\nu_1)\sin(c)\cos(c), \\ R\sin(a) + S\cos(a) = \sin(b)\left(\lambda_1(\mu_1-\nu_1)(\cos(c))^2 + \nu_1(\lambda_1-\mu_1)\right) \end{cases} \tag{3.1}$$

and if $\cos(b) = 0$ then

$$z = \begin{cases} \frac{1}{2}\sin(2(c-a))(\lambda_2-\mu_2)\,\lambda_1(\mu_1-\nu_1) & \text{if} \quad \sin(b) = 1, \\ \frac{1}{2}\sin(2(c+a))(\lambda_2-\mu_2)\,\lambda_1(\mu_1-\nu_1) & \text{if} \quad \sin(b) = -1. \end{cases} \tag{3.2}$$

Suppose that $(0 < \nu_1 < \mu_1 < \lambda_1, 0 < \nu_2 < \mu_2 < \lambda_2)$. Then ξ is harmonic if and only if

$$R\cos(b) = S\cos(b) = z = 0.$$

We distinguish two cases:

• $\cos(b) = 0$. Then ξ is is harmonic if and only if $z = 0$ and, by virtue of (3.2), we get the desired result.

• $\cos(b) \neq 0$ then from (3.1) $\sin(b) = 0$ and $\sin(c)\cos(c) = 0$ and one can check easily that ξ is harmonic if and only if $\cos(a)\sin(a) = 0$.

Except the last case, all the other cases can be deduced in the same way. Let us complete the proof by treating the last case. We suppose that $(0<\nu_1<\mu_1=\lambda_1, 0<\nu_2=\mu_2<\lambda_2)$. Then

$$\begin{cases}\tau(\xi)=\dfrac{\cos(b)\cos(c)(\lambda_1-\nu_1)(\lambda_2-\mu_2)R_1}{\mu_2\lambda_1\nu_1}X_2-\dfrac{2(\lambda_1-\nu_1)(\lambda_2-\mu_2)S_1}{\mu_2\lambda_1\nu_1}X_3,\\ R_1=\sin(a)\sin(c)+\cos(a)\sin(b)\cos(c),\\ S_1=\sin(b)(\cos(a))^2\sin(c)\cos(c)+\dfrac{1}{2}\sin(a)\left(1+\left((\cos(b))^2-2\right)(\cos(c))^2\right)\cos(a)-\dfrac{1}{2}\sin(b)\sin(c)\cos(c)\end{cases}$$

If $\cos(b)=0$ then

$$S_1=\begin{cases}\frac{1}{4}\sin(2(c-a)) & \text{if} \quad \sin(b)=1,\\ -\frac{1}{4}\sin(2(c+a)) & \text{if} \quad \sin(b)=-1\end{cases}$$

and we get (i) and (ii).

If $\cos(c)=0$ then $S_1=\frac{1}{4}\sin(2a)$ and we get (iii).

If $\sin(b)=\sin(c)=0$ the $S_1=R_1=0$ and hence ξ is harmonic.

Suppose now that $\cos(b)\neq 0$, $\cos(c)\neq 0$ and $(\sin(b),\sin(c))\neq(0,0)$. Then ξ is harmonic if and only if $R_1=S_1=0$. We have

$$R_1=\sin(a)\frac{\sin(c)}{\sqrt{\sin^2(c)+\sin^2(b)\cos^2(c)}}+\cos(a)\frac{\sin(b)\cos(c)}{\sqrt{\sin^2(c)+\sin^2(b)\cos^2(c)}}=\sin(a+\alpha)$$

where

$$\cos(\alpha)=\frac{\sin(c)}{\sqrt{\sin^2(c)+\sin^2(b)\cos^2(c)}} \quad \text{and} \quad \sin(\alpha)=\frac{\sin(b)\cos(c)}{\sqrt{\sin^2(c)+\sin^2(b)\cos^2(c)}}.$$

So $R_1=0$ if and only if $a+\alpha=k\pi$ where $k\in\mathbb{Z}$. Thus

$$\cos(a)=(-1)^k\cos(\alpha)=(-1)^k\frac{\sin(c)}{\sqrt{\sin^2(c)+\sin^2(b)\cos^2(c)}} \quad \text{and}$$

$$\sin(a)=-(-1)^k\sin(\alpha)=(-1)^{k+1}\frac{\sin(b)\cos(c)}{\sqrt{\sin^2(c)+\sin^2(b)\cos^2(c)}}.$$

If we replace $\cos(a)$ and $\sin(a)$ in S_1, we get $S_1=0$ which completes the proof. □

The situation for biharmonic homomorphisms is more complicated. We have the following non trivial biharmonic homomorphism which is not harmonic.

Example 24. *The homomorphism* $\xi=\xi_3(a)\circ\xi_2(b)\circ\xi_1(c):(\mathfrak{su}(2),\mathrm{diag}(\lambda_1,\mu_1,\nu_1))\longrightarrow(\mathfrak{su}(2),\mathrm{diag}(\lambda_2,\mu_2,\nu_2))$ *is biharmonic not harmonic if*

$$\mu_1=\nu_1, \mu_2=\nu_2 \quad \textit{and} \quad \cos(a)=\cos(b)=\left(\frac{1}{2}\right)^{\frac{1}{4}}.$$

3.5 Harmonic and biharmonic homomorphisms of sl$(2,\mathbb{R})$

Any homomorphism of sl$(2,\mathbb{R})$ is an automorphism and it is a product $\xi_3(a)\circ\xi_2(b)\circ\xi_1(c)$ where

$$\xi_1(a)=\begin{pmatrix}1&0&0\\0&\cosh(a)&\sinh(a)\\0&\sinh(a)&\cosh(a)\end{pmatrix},\xi_2(a)=\begin{pmatrix}\cosh(a)&0&\sinh(a)\\0&1&0\\\sinh(a)&0&\cosh(a)\end{pmatrix},\xi_3(a)=\begin{pmatrix}\cos(a)&\sin(a)&0\\-\sin(a)&\cos(a)&0\\0&0&1\end{pmatrix}.$$

If $\xi_i:(\mathfrak{su}(2),\langle\ ,\ \rangle_1)\longrightarrow(\mathfrak{su}(2),\langle\ ,\ \rangle_2)$ with $\langle\ ,\ \rangle_j=\mathrm{Diag}(\lambda_j,\mu_j,\nu_j)$ then

$$\begin{cases}\tau(\xi_1(a))=-\dfrac{\cosh(a)\sinh(a)(\mu_2+\nu_2)(\nu_1+\mu_1)}{\lambda_2\mu_1\nu_1}X_1,\\ \tau_2(\xi_1(a))=-2\,\dfrac{(\mu_2+\nu_2)^2\left((\cosh(a))^2-1/2\right)(\nu_1+\mu_1)^2\cosh(a)\sinh(a)}{{\mu_1}^2{\nu_1}^2{\lambda_2}^2}X_1\\ \tau(\xi_2(a))=\dfrac{\cosh(a)\sinh(a)(\lambda_2+\nu_2)(\nu_1+\lambda_1)}{\mu_2\lambda_1\nu_1}X_2,\\ \tau_2(\xi_2(a))=\dfrac{\left(2\,(\cosh(a))^2-1\right)\cosh(a)(\lambda_2+\nu_2)^2(\nu_1+\lambda_1)^2\sinh(a)}{{\lambda_1}^2{\nu_1}^2{\mu_2}^2}X_2,\\ \tau(\xi_3(a))=-\dfrac{\sin(a)\cos(a)(\lambda_2-\mu_2)(-\mu_1+\lambda_1)}{\nu_2\lambda_1\mu_1}X_3,\\ \tau_2(\xi_3(a))=-2\,\dfrac{\sin(a)\cos(a)(-\lambda_2+\mu_2)^2(\mu_1-\lambda_1)^2\left((\cos(a))^2-1/2\right)}{{\lambda_1}^2{\mu_1}^2{\nu_2}^2}X_3.\end{cases}$$

So we get:

Proposition 3.5.1. *1. $\xi_1(a)$ is biharmonic if and only if it is harmonic if only if $a=0$, i.e., $\xi_1=\mathrm{Id}$.*

2. $\xi_2(a)$ is biharmonic if and only if it is harmonic if only if $a=0$, i.e., $\xi_2=\mathrm{Id}$.

3. If $\lambda_2=\mu_2$ or $\lambda_1=\mu_1$ then $\xi_3(a)$ is harmonic.

4. If $\lambda_2\neq\mu_2$ and $\lambda_1\neq\mu_1$ then $\xi_3(a)$ is harmonic if and only if $(\sin(2a)=0)$ and $\xi_3(a)$ is biharmonic not harmonic if and only if $\cos(a)^2=\frac{1}{2}$.

Theorem 3.5.1. *The automorphism*

$$\xi=\xi_3(a)\circ\xi_2(b)\circ\xi_1(c):(\mathrm{sl}(2,\mathbb{R}),\mathrm{diag}(\lambda_1,\mu_1,\nu_1)\longrightarrow(\mathrm{sl}(2,\mathbb{R}),\mathrm{diag}(\lambda_2,\mu_2,\nu_2),0<\lambda_i\leq\mu_i,\nu_i>0$$

is harmonic if and only if $\xi_2(b)=\xi_1(c)=\mathrm{Id}_{\mathrm{sl}(2,\mathbb{R})}$ and $\xi_3(a)$ is harmonic.

Proof. We have

$$\begin{cases} \tau(\xi) = \dfrac{(\mu_2+\nu_2)R}{\lambda_2\lambda_1\mu_1\nu_1}X_1 + \dfrac{(\lambda_2+\nu_2)S}{\mu_2\lambda_1\mu_1\nu_1}X_2 + \dfrac{(\lambda_2-\nu_2)Q}{\nu_2\lambda_1\mu_1\nu_1}X_3, \\ R = \cosh(b)\big(\sinh(b)\,\lambda_1\sin(a)\,(\mu_1+\nu_1)\,(\cosh(c))^2 - \sinh(c)\,\lambda_1\cos(a)\,(\mu_1+\nu_1)\cosh(c) - \sinh(b)\,\nu_1\sin(a)\,(\\ S = \cosh(b)\big(\sinh(b)\,\lambda_1\cos(a)\,(\mu_1+\nu_1)\,(\cosh(c))^2 + \sinh(c)\,\lambda_1\sin(a)\,(\mu_1+\nu_1)\cosh(c) - \sinh(b)\,\nu_1\cos(a)\,(\\ Q = -2\,\cosh(c)\sinh(b)\sinh(c)\,\lambda_1\,(\mu_1+\nu_1)\,(\cos(a))^2 + \cosh(c)\sinh(b)\sinh(c)\,\lambda_1\,(\mu_1+\nu_1) \\ +\sin(a)\left(\lambda_1\left((\cosh(b))^2-2\right)(\mu_1+\nu_1)\,(\cosh(c))^2 - \nu_1\,(\lambda_1-\mu_1)\,(\cosh(b))^2 + \lambda_1\,(\mu_1+\nu_1)\right)\cos(a)\,. \end{cases}$$

On the other hand, one can show easily

$$\begin{cases} \cos(a)R - \sin(a)S = -\cosh(b)\sinh(c)\cosh(c)\,\lambda_1\,(\mu_1+\nu_1)\,, \\ \sin(a)R + \cos(a)S = \left(\lambda_1\,(\mu_1+\nu_1)\,(\cosh(c))^2 + \nu_1\,(\mu_1-\lambda_1)\right)\cosh(b)\sinh(b)\,. \end{cases}$$

So ξ is harmonic if and only if

$$\sinh(b) = \sinh(c) = Q = 0$$

and we get the desired result. $\square$

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